Abstract
In this essay I have tried to present the material on the physics behind the derivation of the Boltzmann Equation. The famous H-Theorem is addressed. The compressible euler equations are got from Boltzmann equation as a fluid dynamic limit. Hilbert’s idea of using the formal power series to find the solution of Boltzmann equation is revisited.

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2 Introduction

The first atomic theory is credited to Democritus of Abdera who lived in the fifth century BC. With rise and fall of the various empires, the concept of atom failed to enter the scientific domain till 18th century. It was with Daniel Bernoulli’s explanation to the gas pressure, the concept of atom entered the scientific domain and the birth of Kinetic theory of gases took place. James Clerk Maxwell and Ludwig Boltzmann revolutionized kinetic theory of gases. Their contribution to the field is immense.

The idea of defining the probability density function on the phase space and studying the evolution of that function is the crux of the Boltzmann Equation. The statistical description of the gas leads to the evolution equation. Newtonian mechanics is employed in studying the interaction between the gas molecules. But, the surprising fact is that the Boltzmann dynamics succeeded in justifying the entropy in the physical systems which the Newtonian description had failed to address. Boltzmann’s H-Theorem in this regard is considered as Boltzmann’s greatest achievements. As we all know, the day today happenings like breaking of glass, ageing etc are all irreversible. If we take the hamiltonian description of all those processes, we can move back in time which is out of reality. Whereas Boltzmann dynamics does not allow you to go backwards. The time is given a direction in this dynamics. Time is said to in increase in that direction along which the H-functional decreases! A very good explanation and analogies regarding the Boltzmann entropy and the time arrow can be found in [6].

The problem of Fluid-dynamic limit is about moving from a Boltzmann description of a dilute gas to a fluid dynamic description, holding on macroscopic scales of space and time. The subject of Fluid-dynamic limit goes back to the work of Maxwell and Boltzmann. A lot of work has been done in this area since 1950.

In this essay, I have shown interest in studying the Boltzmann Equation on Fluid—Dynamic scaling. The scalings that are of interest are Hyperbolic Scaling and Parabolic Scaling. I have dealt with Hyperbolic scaling in this essay. Caflisch’s result is being presented which says that for a smooth solution \((\rho, \theta, u)\) of compressible Euler Equations with \(\rho\) and \(\theta\) bounded away from zero, there exists a solution for hyperbolically rescaled Boltzmann Equation which differs from the local Maxwellian of parameters \((\rho, \theta, u)\) by an infinitesimal of the order of the mean free path. By taking the moments of the local Maxwellian, we can approximate the Euler System. Getting Compressible Euler Equations thus is presented in this essay. Reader who is interested in the detailed description of Caflisch’s result can refer to [5].
3 Deriving Boltzmann Equation

In this section, we shall try to derive the Boltzmann equation in detail. We shall use the notation \(x, dx\) to mean a volume element centered at \(x\) in the particle space and \(\xi, d\xi\) to mean a volume element centered at \(\xi\) in the velocity space. Let us consider a gas in which each molecule is subjected to an external force equal to \(mF\), where \(m\) is the mass of the molecule and \(F\) the force. Without loss of generality we assume \(m=1\). At a time instant \(t\), let us consider a molecule of velocity \(\xi\) at a position \(x\). After a lapse of time equal to \(dt\), the velocity of that molecule would be \(\xi + Fdt\) and its position vector would change to \(x + \xi dt\). If \(f\) is the velocity distribution function, then there are \(f(\xi, x, t)d\xi dx\) molecules which at time \(t\) lie in the volume element \(x, dx\) and have velocities in the range \(\xi, d\xi\). If there were no collisions during that time \(dt\) then the same number of molecules shall be present in the second set that occupy the volume \(x + \xi dt, dx\) with velocities in the range \(\xi + Fdt, d\xi\). The number of molecules in this second set is given by \(f(\xi + Fdt, x + \xi dt, t + dt)d\xi dx\). However, in general the number in the second set will differ from that in the first set, since molecular encounters would have deflected the molecules in the initial set from their course, and will have deflected the other molecules so that they become members of the final set.

The net gain of molecules to the second set is proportional to \(d\xi dx dt\)

\[\Rightarrow \text{The net gain} = \left(\frac{\delta f}{\delta t}\right) d\xi dx dt\]

Hence \([f(\xi + Fdt, x + \xi dt, t + dt) - f(\xi, x, t)]d\xi dx = \left(\frac{\delta f}{\delta t}\right) d\xi dx dt\).

Dividing the above equation by \(d\xi dx dt\) and then tending \(dt\) to zero would give us (using Newton’s Second law)

\[\left(\frac{\delta f}{\delta t}\right) + u(\frac{\delta f}{\delta x}) + v(\frac{\delta f}{\delta y}) + w(\frac{\delta f}{\delta z}) + F_x(\frac{\delta f}{\delta u}) + F_y(\frac{\delta f}{\delta v}) + F_z(\frac{\delta f}{\delta w}) = \left(\frac{\delta f}{\delta t}\right)\]

(1)

where \(\xi = (u, v, w)\) \(\quad F = (F_x, F_y, F_z)\).

The above equation can be written in a compact form as

\[\left(\frac{\delta f}{\delta t}\right) + \nabla_x f + F \cdot \nabla_\xi f = \left(\frac{\delta f}{\delta t}\right)\]

(2)

(2) is said to be the Boltzmann Equation.

3.1 The Collision Term

The quantity \(\left(\frac{\delta f}{\delta t}\right)\) is equal to the rate of change in \(f\) at a fixed point owing to encounters and is called the Collision term. It appears later that the Collision
term can be expressed as an integral involving f itself. The Boltzmann Equation is an Integro-Differential Equation.

Exact expression for the Collision term can be given only when the nature of the interaction between the molecules at encounter is known.

Before going into the details of computing the Collision term, we need to make some simplifying assumptions on the physical systems and the laws of interaction so that the mathematical treatment gets easier.

Some of the major assumptions we make in this article are:

1.-Gas molecules are spherically symmetric.
2.- Force between the molecules is always repulsive at a close encounter.
3.-The intermolecular forces are along the centers of the molecules and hence spherically symmetric.
4.-External force $F$ is negligible and hence taken to be zero.
5.-The velocities of the colliding molecules are assumed to be statistically independent. Any possible correlations between the velocity and position of any single molecule are neglected. This is called the assumption of MOLECULAR CHAOS.

### 3.2 Binary Collision

Let us try to analyze a binary collision. Let A, B be two molecules with initial positions $\mathbf{r}_1$, $\mathbf{r}_2$ respectively and initial velocities before collision $\xi, \xi^*$ respectively. let $m_1$ and $m_2$ be the masses of A and B respectively. let $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ be the relative position of B w.r.t A. The above two molecules move under each other’s influence. The equations of motions of the two are given by

$$m_1 \frac{d^2 \mathbf{r}_1}{dt^2} = \mathbf{F}_{12}$$

$$m_2 \frac{d^2 \mathbf{r}_2}{dt^2} = \mathbf{F}_{21}$$

where $\mathbf{F}_{12}$ and $\mathbf{F}_{21}$ are the force experienced by A due to B and viceversa respectively. We also know that $\mathbf{F}_{12} = - \mathbf{F}_{21}$

Let the velocities of A and B after collision be $\xi^\prime$ and $\xi^\prime*$, let $g_{21}' = \xi^\prime - \xi^*$ be the relative velocity of B with respect to A after collision and $g' = |g_{21}'|$

$$m_0 = m_1 + m_2, M_1 = \frac{m_1}{m_0}, M_2 = \frac{m_2}{m_0}$$

By law of conservation of momentum, we have

$$m_1 \xi + m_2 \xi^* = m_1 \xi^\prime + m_2 \xi^* = m_0 \mathbf{G} \quad (3)$$

Where the constant, $\mathbf{G}$ is the velocity of the center of mass of the above system.

Let us try to evaluate the relative acceleration of the above two body system.

$$\frac{d^2 \mathbf{r}}{dt^2} = \frac{d^2 \mathbf{r}_2}{dt^2} - \frac{d^2 \mathbf{r}_1}{dt^2} = \left( \frac{1}{m_1} + \frac{1}{m_2} \right) \mathbf{F}_{21}$$
Since the force field is assumed to be conservative and along the centers it follows that
\[ F_{21} = -\frac{dU(r)}{dr}e_r \]
where \( e_r \) is a unit vector along \( r \) and \( U(r) \) is the potential of the above force. Hence we have,
\[ m^* \frac{d^2 r}{dt^2} = -\frac{dU(r)}{dr}e_r \]  
(4)
where \( m^* = \frac{m_1 m_2}{m_1 + m_2} \) is the reduced mass of the particles. (4) suggests that analyzing motion of the bodies A and B is same as analyzing the motion of a single body of mass \( m^* \) in a central force field characterized by a potential \( U(r) \). Hence in the next section we start analyzing the motion of a particle in a central force field.

### 3.3 Motion of particle in central force field

A central force is always directed towards a point. We can choose the center of the force as the origin of the coordinate system. Force vector is either parallel or anti-parallel to the radius vector. At any instant, the radius vector and the velocity vector of particle are included in a plane. Since the force vector lies in this plane, the particle has no acceleration normal to this plane. This means that the particle velocity and the force vector always remain in the same plane and the particle trajectory will be confined to this plane. In other words, a particle moving under the influence of a cental force field exhibits a planar motion.

Let the molecule A be taken as the origin of the polar coordinate system. Then the molecule B is moving towards A with the initial relative velocity \( g_{21} \). Let \( r \) and \( \phi \) be the polar coordinates associated with the particle B in the plane of motion. The cartesian coordinates are related to \( r \) and \( \phi \) as follows
\[ x = r \cos \phi \quad y = r \sin \phi \]
(5)
Since the molecule B moves along the trajectory confined to this plane, the molecules motion can be fully described by the time dependent variables \( r(t) \) and \( \phi(t) \). The cartesian velocity components is obtained by differentiating (5) with respect to \( t \).
\[ \dot{x} = \dot{r} \cos \phi - r \dot{\phi} \sin \phi \quad \dot{y} = \dot{r} \sin \phi + r \dot{\phi} \cos \phi \]
(6)
Similarly we can get cartesian acceleration components:
\[ \ddot{x} = \ddot{r} \cos \phi - 2 \dot{r} \dot{\phi} \sin \phi - r \ddot{\phi}^2 \cos \phi - r \dot{\phi} \dot{\phi} \sin \phi \]
\[ \ddot{y} = \ddot{r} \sin \phi + 2 \dot{r} \dot{\phi} \cos \phi - r \ddot{\phi}^2 \sin \phi + r \dot{\phi} \dot{\phi} \cos \phi. \]
(7)
Let $e_x$, $e_y$ represent the orthonormal directions along the x and y directions respectively. Let $e_r$, $e_\phi$ represent the orthonormal vectors in the polar plane. We shall express the initial relative velocity $g_{21}$ as the linear combination of the above mentioned orthonormal vectors.

$$g_{21} = g_x e_x + g_y e_y = g_r e_r + g_\phi e_\phi.$$  

We shall try to evaluate $g_r$ and $g_\phi$ by taking inner product of $(\dot{x}, \dot{y})$ with $e_r$ and then with $e_\phi$.  

$$g_r(e_r, e_r) = (\dot{x}, \dot{y}) (\cos \phi, \sin \phi) = \dot{x} \cos \phi + \dot{y} \sin \phi. \quad (8)$$

$$g_\phi(e_\phi, e_\phi) = (\dot{x}, \dot{y}) (-\sin \phi, \cos \phi) = -\dot{x} \sin \phi + \dot{y} \cos \phi. \quad (9)$$

Similarly, the radial and azimuth component of the acceleration vector can also be got.

$$a_r = \ddot{x} \cos \phi + \ddot{y} \sin \phi \quad a_\phi = -\ddot{x} \sin \phi + \ddot{y} \cos \phi. \quad (10)$$

Substituting (6) in (8) and (9) yields:

$$g_r = \dot{r} \quad g_\phi = r \dot{\phi}. \quad (11)$$

Similarly one can evaluate $a_r$ and $a_\phi$:

$$a_r = \ddot{r} - r \dot{\phi}^2 \quad a_\phi = 2\dot{r} \dot{\phi} + r \ddot{\phi}. \quad (12)$$

Substituting the above radial and azimuth components of acceleration in (4), we get

$$m^* a_r = m^* (\ddot{r} - r \dot{\phi}^2) = -\frac{dU(r)}{dr}$$

$$m^* a_\phi = m^* (2\dot{r} \dot{\phi} + r \ddot{\phi}) = 0. \quad (13)$$

This shows that the acceleration is only along the radial direction and the azimuth acceleration is identically zero. the angular momentum vector of this relative motion is given by

$$L = \mathbf{r} \times m^* g_{21}\quad (14)$$

Since, both $\mathbf{r}$ and $\mathbf{g}_{21}$ are in the plane of motion, $L$ is perpendicular to the plane of motion. Hence, $L = L \mathbf{e}_n$, where $\mathbf{e}_n$ is a unit normal vector to the reference plane. We know by the definition of Angular momentum that the component, $L$ is equal to:

$$L = m^* r^2 \dot{\phi}\quad (15)$$

Differentiating (15) with respect to $t$ will yield:

$$\dot{L} = 2 m^* r \dot{r} \dot{\phi} + m^* r^2 \ddot{\phi} = m^* a_\phi = 0.$$
Since $e_n$ remains constant, we have

$$\dot{L} = 0$$  \hspace{1cm} (16)$$

(16) implies that in a binary collision there are no external torques applied to the system and therefore the angular momentum of the relative motion is conserved. From (13) we have,

$$m^*a_r + \frac{dU(r)}{dr} = 0$$

$$m^*(i - r\dot{\phi})^2 + \frac{dU(r)}{dr} = 0$$

$$m^*\ddot{r} - \frac{L^2}{m^*r^3} + \frac{dU(r)}{dr} = m^*\dot{r} + \frac{d}{dr} \left\{ \frac{L^2}{2m^*r^2} + U \right\} = 0$$

Multiplying by $\frac{dr}{dt}$ and integrating with respect to $t$, yields

$$\int dt \left\{ m^*\dot{r} + \frac{d}{dr} \left\{ \frac{L^2}{2m^*r^2} + U \right\} \right\} = 0.$$ 

$$\int m^*\dot{r}d\dot{r} + \int dr \frac{d}{dr} \left\{ \frac{L^2}{2m^*r^2} + U \right\} = 0.$$ 

This leads to a conservation law:

$$\frac{1}{2}m^*\dot{r}^2 + \frac{L^2}{2m^*r^2} + U = \text{constant}$$  \hspace{1cm} (17)$$

(17) express the conservation of energy for the relative motion of the two particles A and B under consideration. Let $g = |g_{21}|$. If there exists no interaction then $g_{21}$ remain constant making the trajectory a straight line. Let us consider a cartesian coordinate system with origin at A and the x-axis anti-parallel to the $g_{21}$. The trajectory depends on the interaction potential $U(r)$, the initial relative velocity $g_{21}$, and the impact parameter $b$ which we shall define now.

**Definition**

The Impact parameter $b$, is the smallest distance between the initial assymptote to the trajectory and the x-axis.(see figure 1)

The initial angular momentum of the relative motion is given by

$$L = m^*gr_0sin\phi = m^*gb.$$ 

Now let us try to evaluate the constant in (17) in the limiting case $r \to \infty$.

$$\lim \left\{ \frac{1}{2}m^*\dot{r}^2 + \frac{L^2}{2m^*r^2} + U \right\} = \frac{1}{2}m^* \left\{ i^2 + r^2\dot{\phi}^2 \right\} \quad \text{[since } U \to 0 \text{ as } r \to \infty \text{]}$$

$$= \frac{1}{2}m^* \left\{ g_r^2 + g_{\phi}^2 \right\}$$

$$= \frac{1}{2}m^* g^2.$$
Thus the relative motion of the two molecules is described by the following two equations:

\[ m^* r^2 \dot{\phi} = m^* gb. \]  

(18)

\[ \frac{1}{2} m^* r^2 + \frac{m^* g^2 b^2}{2 r^2} + U(r) = \frac{1}{2} m^* g^2. \]  

(19)

We shall try to remove the time dependence in the above equations and try to express r as a function of \( \phi \).

\[ (d\phi)^2 = \frac{g^2 b^2}{r^4} (dt)^2 \]

\[ (dr)^2 = \frac{g^2}{r^2} \left\{ r^2 - b^2 - \frac{2r^2 U(r)}{m^* g^2} \right\} (dt)^2. \]

Dividing one by the other we get,

\[ \frac{dr}{d\phi} = \pm \frac{r}{b} \sqrt{r^2 - b^2 - \frac{2r^2 U(r)}{m^* g^2}} \]  

(20)

The negative sign in (20) corresponds to the first portion of the trajectory where B approaches A (r decreasing with increasing \( \phi \)) and the positive sign corresponds to the second portion of the trajectory where B moves away from A (r increasing with increasing \( \phi \))(see figure). Let \( r_m \) and \( \phi_m \) represent the radial and azimuth component of the closest approach(see figure 1). Since \( \frac{dr}{d\phi} < 0 \) for \( \phi < \phi_m \) and \( \frac{dr}{d\phi} > 0 \) for \( \phi > \phi_m \), we have \( \frac{dr}{d\phi} = 0 \) at \( \phi = \phi_m \). This implies that,

\[ r_m^2 - b^2 - \frac{2r_m^2 U(r_m)}{m^* g^2} = 0 \]  

(21)

The equation

\[ r^2 - b^2 - \frac{2r^2 U(r)}{m^* g^2} = 0 \]  

(22)

may have multiple solutions. We shall consider \( r_m \) to be the maximum of all the solutions of this equation. For the (20) to make sense, the LHS of (22) should be non negative for \( r \geq r_m \) as the molecule B cannot have radius vector \( r < r_m \). Since the LHS of (22) tends to infinity as \( r \) tends to infinity, we know that the term inside the square root sign remains positive for all \( r \geq r_m \) as \( r_m \) is the maximum of all the zeroes of the LHS of (22). Hence all repulsive potentials are allowed.

Now let us try to integrate (20) to get an implicit expression for \( r(\phi) \).

\[ \phi - \phi_m = \pm \int_{r_m}^{r} \frac{b}{r} \left\{ r^2 - b^2 - \frac{2r^2 U(r)}{m^* g^2} \right\}^{-\frac{1}{2}} dr. \]  

(23)
Let us try to get an expression for $\phi_m$. We know that well before the particles start interacting i.e, $r = \infty$, we have $\phi = 0$. Hence

$$\phi_m = \int_{r_m}^{\infty} \frac{b}{r} \left\{ r^2 - b^2 - \frac{2r^2U(r)}{m^*g^2} \right\}^{-\frac{1}{2}} \, dr. \quad (24)$$

Since the force field is symmetric, trajectory is symmetric along the closest approach, the azimuth angle of the outgoing asymptote, $\phi_o = 2\phi_m$ (see figure 1). Hence the deflection angle, $\chi = \pi - 2\phi_m$. This implies,

$$\chi = \pi - 2 \int_{r_m}^{\infty} \frac{b}{r} \left\{ r^2 - b^2 - \frac{2r^2U(r)}{m^*g^2} \right\}^{-\frac{1}{2}} \, dr. \quad (25)$$

(25) shows that $b$ and $\chi$ are physically equivalent quantities. It provides a unique transformation between $b$ and $\chi$ for a given intermolecular potential and initial relative velocity. We shall try to show that $\chi(b)$ is a monotonically decreasing function.

Apply change of variable by $x = \frac{r}{r_m}$ in (25).

$$\chi = \pi - 2 \int_{r_m}^{\infty} \frac{b}{r} \int_{1}^{\infty} \frac{dx}{x \sqrt{x^2 - \frac{b^2}{r_m^2} - \frac{2x^2U(x)}{m^*g^2}}} \, dx$$

Differentiating the above equation with respect to $b$, we get:

$$\frac{d\chi}{db} = -2 \left\{ \frac{d}{db} \left( \frac{b}{r_m} \right) \int_{1}^{\infty} \frac{dx}{x \sqrt{x^2 - \frac{b^2}{r_m^2} - \frac{2x^2U(x)}{m^*g^2}}} + \frac{d}{db} \left( \frac{b}{r_m} \right) \int_{1}^{\infty} \frac{dx \frac{b^2}{r_m^2}}{x \left[ x^2 - \frac{b^2}{r_m^2} - \frac{2x^2U(x)}{m^*g^2} \right]^2} \right\}. \quad (26)$$

It follows from the analysis of (22) done earlier that the denominator in the integrand of (26) is positive for $x > 1$. Hence the numerator in the integrand also remains positive for $x > 1$. Therefore, the numerator in the integrand also remains positive for $x > 1$. Hence, we have shown that $\frac{d\chi}{db}$ is negative definite. Hence, $\chi(b)$ is a monotonous decreasing function of $b$. Hence the relation between $\chi$ and $b$ is one-one.
An interesting observation can be made if a constant potential case is considered with \( U(r) = U \). Then \( r_m \) can be expressed as

\[
  r_m = \frac{b}{\sqrt{1 - \frac{2U}{m^*g^2}}},
\]

Hence, \( \frac{d}{db} \left( \frac{b}{r_m} \right) = 0 \) which means that \( \frac{d\chi}{db} = 0 \).

Let us now try to calculate the deflection angle \( \chi \) in some familiar models. Let us consider a very fast relative motion so that \( \frac{2U}{m^*g^2} \) can be neglected everywhere along the trajectory. In this case \( r_m = b \) i.e, the trajectory is a straight line.

\[
  \chi = \pi - 2 \int_b^\infty \frac{b \, dr}{r\sqrt{r^2 - b^2}} = \pi - 2 \left\{ \arccos \left( \frac{b}{r} \right) \right\}_b^\infty = 0.
\]

as expected.

### 3.4 Hard Sphere Model

The assumptions of this model are:
1. The gas molecules are thought to be hard elastic spheres.
2. Spheres are perfectly impenetrable.
3. Spheres have no effect on one another unless their surfaces are in contact.
4. Intermolecular potential is zero when their surfaces are not in contact.

If \( d_1 \) and \( d_2 \) are diameters of A and B respectively then the distance between their centers when their surfaces touch is given by \( d = \frac{(d_1+d_2)}{2} \).

In this case,

\[
  \chi = \pi - 2 \int_d^\infty \frac{b \, dr}{r\sqrt{r^2 - b^2}} = \pi - 2 \left\{ \arccos \left( \frac{b}{r} \right) \right\}_d^\infty.
\]

This implies that

\[
  b = d \cos \left( \frac{\chi}{2} \right).
\]

### 3.5 Collision Cross section

In physical applications we are usually concerned not with the deflection of a single molecule but with the scattering of a beam of identical molecules incident with uniform velocity \( g_21 \) on the scattering centre. Different molecules in the beam have different impact parameter and therefore will be scattered through different angles. Let the intensity of the incident beam be equal to \( n \). Some part of the incident beam gets scattered into a unit solid angle \( d\Omega \). The number
of particles scattered into this $d\Omega$ is proportional to $n, d\Omega$. Hence, the number particles scattered in unit time is

$$dN = S(g_{21}, g'_{21}) \ n \ d\Omega.$$  

The proportionality constant $S(g_{21}, g'_{21})$ has got the same dimension as that of area and it is called Differential Cross Section. The physical interpretation of $S(g_{21}, g'_{21})$ is that it is a area perpendicular to $g_{21}$ which scatters molecules with relative velocity $g_{21}$ into an infinitesimal solid angle around $g'_{21}$. Total particles scattered, $N$ can be got by integrating the above equation over all possible directions of $g'_{21}$ i.e,

$$N = n \int d\Omega \ S(g_{21}, g'_{21}).$$

**Definition**

Total scattering Cross section is defined by the following integral

$$\sigma(g, g') = \int d\Omega \ S(g_{21}, g'_{21}). \ (28)$$

Consider a plane which is perpendicular to $g_{21}$ and contain the molecule $A$ (say Plane 1). Let us fix an arbitrary direction in that plane. The impact parameter $b$ and the deflection angle $\chi$ are measured in the plane that contains the trajectory. The plane of motion cuts the Plane 1 along a line. The angle $\varepsilon$ between this line of intersection and the fixed line is called the azimuth angle. $\varepsilon$ can vary from 0 to $2\pi$ and $\chi$ can vary from 0 to $\pi$. $b, \chi, \varepsilon$ serve as the spherical coordinates. Hence, we can express the unit solid angle $d\Omega$ as

$$d\Omega = \sin \chi \ d\chi \ d\varepsilon.$$  

Due to the spherical symmetry of the central force field, the differential cross section, $S$, must be independent of the azimuth angle, $\varepsilon$, and the direction of the initial relative velocity, $g_{21}$. This means that $S = S(g, \chi)$. In case of spherically symmetric force fields, there exists an axis of symmetry parallel to $g_{21}$ and goes through the center of the target molecule. We also know that there is a one-to-one correspondance between $\chi$ and $b$. Hence the molecules approaching the target along a cylinder of radius $b$ centered at the axis of symmetry will have the same deflection angle $\chi$. The area element perpendicular to $g_{21}$ that scatters molecules with deflection angles between $\chi$ and $\chi + d\chi$ is given by $2 \pi b \ db$ and the area which scatters particles with deflection angles between $\chi$ and $\chi + d\chi$ and with azimuth angles between $\varepsilon$ and $\varepsilon + d\varepsilon$ is $b \ db \ d\varepsilon$. This area element is exactly the differential cross section. hence,

$$S(g, \chi) \ \sin \chi \ d\chi \ d\varepsilon = b \ db \ d\varepsilon.$$
\[ S(g, \chi) = \frac{b(\chi)}{\sin \chi} |\frac{db}{d\chi}|. \] (29)

Since we need \( S(g, \chi) \) to be positive we use the modulus of the derivative \( \frac{db}{d\chi} \) as the derivative may be negative. Let us try to compute the cross sections in hard sphere case. Differentiating (27) with respect to \( \chi \) gives

\[ \frac{db}{d\chi} = -\frac{d}{2} \sin(\frac{\chi}{2}) \]

Substituting the above expression in the (29) yields the following expression for the differential cross section for the hard sphere molecule.

\[ S(g, \chi) = \frac{d^2}{4}. \] (30)

The total scattering cross section can be got by integrating (30) over the entire solid angle space:

\[
\sigma(g) = \int_{0}^{\pi} d\chi \sin \chi \int_{0}^{2\pi} d\varepsilon S(g, \chi) = \int_{0}^{\pi} d\chi \sin \chi \int_{0}^{2\pi} d\varepsilon \frac{d^2}{4} = \pi d^2.
\] (31)

This result simply tells us that the cross section of a hard sphere molecule is the cross section of the sphere of influence. This is not surprising and is completely consistent with the physical intuition.

### 3.6 Statistics of the molecular encounter

In this sub section, we will try to find out the collisin term in the Boltzmann Equation. Our goal is to calculate the net change in the number of molecules inside \( x, dx \) with velocity vectors in \( \xi, d\xi \). The net change is result of two competing processes. First, the molecules in \( x, dx \) with velocities in range \( \xi, d\xi \) may get scattered due to collision with other molecules. This is the loss term in the collision term. Second, the molecules in \( x, dx \) whose velocities are not in \( \xi, d\xi \) can be scattered into our specified velocity range. This count will give the gain term. The collision term can be expressed as the difference of the gain and the loss terms.

At first, we focus our attention on the loss term. By A-like molecules we mean the molecules in \( x, dx \) whose velocities lie in the range \( \xi, d\xi \) and by B-like molecules we mean the molecules in \( x, dx \) whose velocities lie in the range \( \xi^*, d\xi^* \). Looking into the Plane 1 that was mentioned in the previous subsection, for the molecule B to encounter with Molecule A, the trajectory should
hit a patch of the area $b\,db\,d\varepsilon$ on the plane 1 as shown in the figure. If we consider a time $dt$ before the trajectory hits the Plane 1, we can say that $B$ is contained in a cylinder with base area $b\,db\,d\varepsilon$ and height $g\,dt$. The volume of that cylinder is $g\,b\,db\,d\varepsilon\,dt$. Similarly we can assign one such cylinder to each A-like molecule i.e., we can have $fd\xi dx$ such cylinders. By $f$ we mean $f(\xi, x, t)$. $db$ and $d\varepsilon$ can be chosen small enough such that the overlapping of the above mentioned cylinders can be neglected. Then the total volume of all such cylinders is

$$dv = f\,b\,g\,d\chi\,db\,d\varepsilon\,dx\,dt.$$  \hspace{1cm} (32)

All the cylinders mentioned above may not contain B-like molecules in them. By choosing $d\xi_*$ small enough we can avoid the possibility of a single cylinder having two B-like molecules. The number of B-like molecules present in the volume $dv$ is given by

$$f_*\,d\xi_*\,dv = f\,f_*\,g\,S(g, \chi)\sin\chi\,d\chi\,d\varepsilon\,dx\,d\xi\,d\xi_*\,dt.$$  \hspace{1cm} (33)

If a A-like molecule encounters a molecule B-like, then its velocity goes out of $\xi$, $d\varepsilon$ i.e., they are lost from the A-like set. We know that the loss is proportional to $d\xi dx dt$. So consider

$$f\,f_*\,g\,S(g, \chi)\sin\chi\,d\chi\,d\varepsilon\,d\xi\,d\xi_*.$$  \hspace{1cm} (34)

Hence by integrating (34) with respect to all possible $c_2$'s in the velocity space and with respect to the whole range of $\varepsilon$ and $\chi$, we get the required loss term.

$$\left\{ \frac{\delta f}{\delta t} \right\}_- = \int d\xi_* \int_0^{2\pi} d\varepsilon \int_0^{\pi} d\chi \sin\chi\,S(g, \chi)\,g\,f\,f_*.$$  \hspace{1cm} (35)

where the first integral is in the velocity space.

Now we turn our attention to computing the gain term. We need to calculate the number of molecules which become A-like after collision. Here, we propound a theory which says that there is an inverse encounter for every forward encounter. It says that, if two molecules with velocities $\xi$ and $\xi_*$ interact to gain final velocities $\xi'$ and $\xi'_*$ then, there is an inverse encounter in which two molecules with velocities $\xi'$ and $\xi'_*$ interact to gain final velocities $\xi$ and $\xi_*$. Hence, the analysis which was done for the computation of the loss term can be done on similar lines to get the gain term.

Let us consider an encounter between the molecules with initial velocities $\xi'$ and $\xi'_*$, with deflection angle $\chi'$ and azimuth angle $\varepsilon'$. Because of the conservation of momentum (3), we have that the final velocities of them would be $\xi$
and $\xi_*$. From the conservation of the angular momentum (17) we have that the impact parameter, $b$, before and after collision are the same. Consequently, the deflection angles of the forward and inverse collisions must be identical, $\chi = \chi'$. since we have shown that the relative motion of the two particles is planar, the azimuth angle of the forward and the inverse collisions are the same, $\varepsilon = \varepsilon'$.

From (3), we have

\[
\begin{align*}
\xi &= G - M_2 \, g_{21} \\
\xi_* &= G + M_1 \, g_{21} \\
\xi' &= G - M_2 \, g'_{21} \\
\xi'_* &= G + M_1 \, g'_{21}
\end{align*}
\]

By law of conservation of energy we have,

\[
\frac{1}{2} \left( m_1 \, \xi^2 + m_2 \, \xi_*^2 \right) = \frac{1}{2} \left( m_1 \, \xi'^2 + m_2 \, \xi'_*^2 \right)
\]

Substituting for $\xi, \xi, \xi', \xi'_*$ in (37) from (36) yields that $g = g'$. An analogue for (33) in case of inverse encounter is given by,

\[
f' f'_* \, g' S(g', \chi') \sin \chi' \, d\xi' \, d\varepsilon' \, dx \, d\xi \, d\xi'_* \, dt.
\]

Where $f'$ and $f'_*$ represent $f(\xi', x, t)$ and $f(\xi'_*, x, t)$ respectively. Since we have shown that $g = g'$, $\chi = \chi'$, $\varepsilon = \varepsilon'$, (38) becomes,

\[
f' f'_* \, g \, S(g, \chi) \sin \chi \, d\xi \, d\varepsilon \, dx \, d\xi' \, d\xi'_* \, dt.
\]

Now we shall try to express $d\xi' \, d\xi'_*$ in terms of $d\xi \, d\xi_*$ so that we can add the gain and the loss terms. To change variables from $d\xi \, d\xi_*$ to $dG \, dg_{21}$ we need to compute the Jacobian.

\[
J = \frac{\delta (G, g_{21})}{\delta (\xi, \xi_*)} = \frac{\delta (\xi + M_2 g_{21}, g_{21})}{\delta (\xi, \xi_*)} = \frac{\delta (\xi, g_{21})}{\delta (\xi, \xi_*)} = \frac{\delta (\xi, \xi_*)}{\delta (\xi', \xi'_*)} = I
\]

Similarly

\[
J' = \frac{\delta (G, g'_{21})}{\delta (\xi', \xi'_*)} = I.
\]
Hence
\[ dG \, dg_{21} = d\xi \, d\xi^* \]  
\[ dG \, dg'_{21} = d\xi' \, d\xi'^* \]  
Equation (40)

We know that
\[ dg_{21} = g^2 \, dg \, d\Omega \]
\[ dg'_{21} = g^2 \, dg \, d\Omega' \]

Where \( d\Omega = d\Omega' = \sin \chi \, d\chi \, d\varepsilon \). Thus,
\[ d\xi' \, d\xi'^* = dG \, dg_{21} \]
\[ = dG \, g^2 \, dg \, d\Omega' \]
\[ = dG \, g^2 \, dg \, d\Omega \]
\[ = dG \, dg_{21} \]
\[ = d\xi \, d\xi^* \]

Substituting the above relation in (39) will lead us to
\[ f' \, f'_* \, g \, S(g, \chi) \sin \chi \, d\chi \, d\varepsilon \, dx \, d\xi \, d\xi^* \, dt. \]  
Equation (41)

Hence the gain term turns out to be
\[ \left\{ \frac{\delta f}{\delta t} \right\}_+ = \int d\xi^* \int_0^{2\pi} d\varepsilon \int_0^\pi d\chi \sin \chi \, S(g, \chi) \, g \, f' \, f'_* . \]  
Equation (42)

With the gain and loss terms in our arsenal, we are in a position to write the needed collision term. Hence the collision term is given by
\[ \left\{ \frac{\delta e}{\delta t} \right\} = \left\{ \frac{\delta f}{\delta t} \right\}_+ - \left\{ \frac{\delta f}{\delta t} \right\}_- \]
\[ = \int d\xi^* \int_0^{2\pi} d\varepsilon \int_0^\pi d\chi \sin \chi \, S(g, \chi) \, g \, [f' \, f'_* - f \, f_*]. \]  
Equation (43)

Substituting (43) in (2) yields the Boltzmann Equation.
\[ \left( \frac{\delta f}{\delta t} \right) + \xi \cdot \nabla_x f + F \cdot \nabla \xi f = \int d\xi^* \int_0^{2\pi} d\varepsilon \int_0^\pi d\chi \sin \chi \, S(g, \chi) \, g \, [f' \, f'_* - f \, f_*]. \]  
Equation (44)

By our assumption \( F = 0 \). Hence (44) will become
\[ \left( \frac{\delta f}{\delta t} \right) + \xi \cdot \nabla_x f = \int d\xi^* \int_0^{2\pi} d\varepsilon \int_0^\pi d\chi \sin \chi \, S(g, \chi) \, g \, [f' \, f'_* - f \, f_*]. \]  
Equation (45)
We also have the relation
\[ \sin \chi d\chi d\varepsilon = d\omega. \]
where \( d\omega \) represents the unit solid angle around the unit vector \( \omega \). Hence (45) can be transformed into an equation with \( d\omega \) in it integrated over whole of unit sphere \( S^2 \). That is,
\[ \frac{\delta f}{\delta t} + \xi \cdot \nabla_x f = \int_{S^2} \int_{\mathbb{R}^3} [f'f'_* - ff_*] b(g, \omega) \, d\omega \, d\xi. \quad (46) \]
where \( b(g, \omega) = g \, S(g, \chi) \). From now onwards we denote the collision term by \( Q(f, f) \). Hence the compact form of Boltzmann equation is given by
\[ \frac{\delta f}{\delta t} + \xi \cdot \nabla_x f = Q(f, f). \quad (47) \]

4 Collision Invariants

In this section we shall try to determine some of the quantities that remain invariant during the collision process. While deriving the Boltzmann Equation, we assumed conservation of momentum and conservation of energy. Thus we already know two collision invariants. Usually, a question arises: Are these the only collision invariants? By a very beautiful analysis, we will be able to prove that any collision invariant is of the form
\[ \phi(\xi) = a + b \cdot \xi + c |\xi|^2. \quad (48) \]
where \( a, c \in \mathbb{R} \), \( b \in \mathbb{R}^3 \).

To find the collision invariants, we would like to solve for \( \phi \) in
\[ \int_{\mathbb{R}^3} \phi(\xi) \, Q(f, f) \, d\xi = 0. \quad (49) \]
The integral equation in (49) says that the average of \( \phi(\xi) \) in the velocity space subjected to collision process is zero. We have
\[ \int_{\mathbb{R}^3} \phi(\xi) \, Q(f, f) \, d\xi = \int_{\mathbb{R}^3} \int_{S^2} [f'f'_* - ff_*] \phi(\xi) \, b(\xi - \xi_*, \omega) \, d\xi_* \, d\omega \, d\xi. \quad (50) \]
Making change of variables \((\xi, \xi_*) \mapsto (\xi_*, \xi)\) keeping the \( \omega \) fixed would result in \((\xi', \xi'_*) \) changing to \((\xi*', \xi')\) due to (3). The RHS of (50) would become
\[ = \int_{\mathbb{R}^3} \int_{S^2} [f'f'_* - ff_*] \phi(\xi_*) \, b(\xi - \xi_*, \omega) \, d\xi_* \, d\omega \, d\xi. \quad (51) \]
Because, as it turns out, the collision kernel \( b(V, \omega) \) is a.e positive and is of the form
\[ b(V, \omega) = |V| \, S(|V|, |\cos((V, W))|). \quad (52) \]
Hence \( b(\xi - \xi_*, \omega) = b(\xi_* - \xi, \omega) \).

In (50), let us make a change of variables given by \((\xi, \xi_*) \mapsto (\xi', \xi'_*)\). In section 2.6 we have shown that \( \frac{d\xi}{d\xi_*} = \frac{d\xi'}{d\xi'_*} \). Thus RHS of (50) becomes

\[
= \int_{\mathbb{R}^3} \int_{S^2} [f f'_* - f f_*] \phi(\xi') \ b(\xi - \xi_*, \omega) \ d\xi_* \ d\omega \ d\xi.
\]

(53)

Because, using (37), we have already shown that \( g = g' \). By the argument that \( b \) depends on \(|V| = g \) implies that \( b(\xi' - \xi'_*, \omega) = b(\xi - \xi_*, \omega) \).

An other cahnage of variable \((\xi', \xi'_*) \mapsto (\xi'_*, \xi')\) in (53) would result in

\[
= \int_{\mathbb{R}^3} \int_{S^2} [f f'_* - f f_*] \phi(\xi'_*) \ b(\xi - \xi_*, \omega) \ d\xi_* \ d\omega \ d\xi.
\]

(54)

Let us add (50), (51), (53), (54) to get

\[
\int_{\mathbb{R}^3} \phi(\xi) Q(f, f) d\xi = \int_{\mathbb{R}^3} \int_{S^2} [f f'_* - f f_*] [\phi(\xi) + \phi(\xi') - \phi(\xi'_*) - \phi(\xi'_*)] \ b(\xi - \xi_*, \omega) \ d\xi_* \ d\omega \ d\xi.
\]

(55)

The RHS of (55) will be zero, independent of \( f \), if

\[
\phi(\xi) + \phi(\xi + \xi_* - \xi) = \phi(\xi) + \phi(\xi + u + v)
\]

(56)

for all \( \xi, \xi_*, \xi', \xi'_* \in \mathbb{R}^3 \) such that \( \xi + \xi_* = \xi'_* + \xi'_* \).

All those measurable functions defined a.e on \( \mathbb{R}^3 \) and satisfy (56) are called collision invariants. We have

\[
\phi(\xi) + \phi(\xi + \xi_* - \xi) = \phi(\xi) + \phi(\xi + u + v)
\]

(57)

where \( \xi_* - \xi = u + v \) and \( u \cdot v = 0 \)

The RHS of (57) is equal to

\[
= \phi(\xi + u) + \phi(\xi + v).
\]

Hence we have

\[
\phi(\xi) + \phi(\xi + u + v) = \phi(\xi + u) + \phi(\xi + v). \text{ whenever } u \cdot v = 0.
\]

(58)

**Lemma:** In order to solve (58), it is enough to solve for \( p \)

\[
p(u + v) = p(u) + p(v). \text{ whenever } u \cdot v = 0.
\]

(59)

**proof:** Suppose \( \exists \phi \) such that \( \phi \) solves (58). Define

\[
p(u) = \phi(\xi + u) - \phi(\xi).
\]
\[ p(u + v) = \phi(\xi + u + v) - \phi(\xi) + \phi(\xi) - \phi(\xi) \]
\[ p(u + v) = \phi(\xi + u) - \phi(\xi) + \phi(\xi + v) - \phi(\xi) \]
\[ p(u + v) = p(u) + p(v). \]

Suppose now that \( \exists \ p \) which solves (59). Then define \( \phi \) implicitly in the form
\[ p(u) = \phi(\xi + u) - \phi(\xi). \]
for some \( u \) arbitrarily fixed. We can proceed as before to show that \( \phi \) solves (58).

Our main objective in this section is to characterize the collision invariants. Before addressing the main theorem of this section, we will give here a theorem which comes in handy during the proof of the main theorem.

**Cauchy’s Theorem:*** If \( p : \mathbb{R}^n \mapsto \mathbb{R} \) and \( p \) is continuous in atleast one point and satisfies
\[ p(x + y) = p(x) + p(y) \]
then \( \exists \) a constant vector \( A \in \mathbb{R}^n \) such that \( p(x) = A \cdot x \) for all \( x \in \mathbb{R}^n \).

**Main Theorem of this section:*** If \( p : \mathbb{R}^n \mapsto \mathbb{R} \) is continuous and satisfies (59) for \( (u, v) \in M = \{u, v \in \mathbb{R}^3 : u \cdot v = 0\} \), then for some \( b \in \mathbb{R}^3, c \in \mathbb{R} \) it holds that
\[ p(\xi) = b \cdot \xi + c |\xi|^2. \]

**Proof:** Let \( p \) be decomposed into a sum of an odd and an even function. i.e,
\[ k(\xi) = \frac{p(\xi) + p(-\xi)}{2} \]
\[ h(\xi) = \frac{p(\xi) - p(-\xi)}{2}. \]
Since \( p \) satisfies (59), both \( k \) and \( h \) satisfy (59). Let us consider \( x, y \in \mathbb{R}^3 \) such that \( |x| = |y| = r \). Let \( u = \frac{x+y}{2}, v = \frac{x-y}{2} \). Trivially, \( u \cdot v = 0 \) As \( k \) is an even function, we have
\[ k(u + v) = k(u) + k(v) = k(u - v) \]
\[ k(x) = K(y). \]
Thus, \( \exists \) a continuous function \( \phi \) such that \( k(x) = \phi(|x|^2) \). Hence we have
\[ \phi(|u|^2) + \phi(|v|^2) = \phi(|u|^2 + |v|^2). \]
Hence by cauchy’s theorem we have the existence of \( c \in \mathbb{R} \) such that \( \phi(|u|^2) = c \cdot |u|^2. \)
Now consider the odd function \( h \). Let \( \{e_i\}_{i=1}^3 \) be the basis of \( \mathbb{R}^3 \). We know that \( (e_i, e_j), \ j \in M \). Let \( (u,v) \in M \). Also \( u = \sum_1^3 u_i \ e_i \) and \( v = \sum_1^3 v_i \ e_i \). Since \( h \) satisfies (58) with \( (u, v) \) we have
\[
\sum_1^3 (h(u_i + v_i)e_i) = \sum_1^3 h(u_i e_i) + \sum_1^3 h(v_i e_i)
\]
whenever \( (u_1, u_2, u_3) \) is perpendicular to \( (v_1, v_2, v_3) \), \( (u_1, -u_2, -u_3) \) is perpendicular to \( (v_1, -v_2, -v_3) \). If the above analysis is carried out on the second pair, we will get the same LHS as the above equation but RHS will be of opposite sign. Thus we conclude that
\[
h(u_1 e_1) + h(v_1 e_1) - h((u_1 + v_1)e_1) = 0.
\]
Similar expression can be got for \( i = 1, 2, 3 \). Here again we apply Cauchy’s theorem to get \( b_i \in \mathbb{R} \) such that
\[
h(u_i e_i) = b_i e_i.
\]
Hence
\[
h(u) = b \cdot u.
\]
From the above analysis we get
\[
p(u) = b \cdot u + c|u|^2.
\]
The above expression for \( p \) is modulo a constant. Hence the general expression for a collision invariant is got to be
\[
p(\xi) = a + b \cdot \xi + c|\xi|^2.
\]

5 Boltzmann Inequality and the H-Theorem

Let us consider the collision integral
\[
Q(f, f) = \int_{S^2} \int_{\mathbb{R}^3} [f' f^* - ff^*] b(g, \omega) \ d\omega \ d\xi^*.
\]
We will try to find out the \( f \)'s that give a vanishing \( Q \).

Before answering the above question, we would like to establish the Boltzmann inequality.

If \( f \) is a non negative function that solves the Boltzmann equation such that \( \log f \) \( Q(f, f) \) is integrable and the manipulations done on \( \phi \) in the previous section hold for \( \phi = \log f \) then,
\[
\int_{\mathbb{R}^3} \log f \ Q(f, f) \ d\xi \leq 0.
\]
Since we have characterized the collision invariants, the equality in (62) holds iff
\( f(\xi) = \exp\{a + b.\xi + c|\xi|^2\} \). Since we have assumed the validity of manipulations in \( \phi \) of the previous section with \( \phi = \log f \), we have

\[
\int_{\mathbb{R}^3} \log f \ Q(f, f) d\xi = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \log \left( \frac{ff_*}{ff'_*} \right) [f'f'_* - ff_*] b(g, \omega) d\omega d\xi d\xi^*
\]

Since \( b \) is a positive quantity, we have to analyze only \( \log \left( \frac{ff_*}{ff'_*} \right) \{1 - \left( \frac{ff_*}{ff'_*} \right) \} \).

Let \( \frac{ff_*}{ff'_*} = x \). We have from elementary calculus that 
\( (1 - x) \log x \leq 0 \) for all \( x > 0 \). Thus we have proved the Boltzmann inequality. The equality is possible only when \( x = 1 \) i.e,
\( ff_* = f'f'_* \)
applies almost everywhere. Taking logarithms on both sides yields that \( \phi = \log f \) is a collision invariant.

\[
f(\xi) = \exp\{a + b.\xi + c|\xi|^2\}
\]

Such \( f \)'s are called the Maxwellians. The above \( f \) can be expressed as

\[
f(\xi) = A \exp\{-\beta |\xi - u|^2\} \quad (63)
\]

The constants \( A, \beta \) shall be addressed in the next section. The term \( u \) is the bulk velocity and is defined in the next section. A good exposition on Maxwellian Distributions can be found in [9].

Let us now turn our attention to the question asked at the beginning of this section. i.e,

\[
Q(f, f) = 0. \quad (64)
\]

Multiplying both sides of (64) by \( \log f \) and integrating with respect to \( d\xi \) results in the equality in (62). That would imply that \( f \) is a Maxwellian.

Thus we have shown that
\[
\int_{\mathbb{R}^3} \log f \ Q(f, f) d\xi = 0
\]
iff
\[
Q(f, f) = 0
\]
iff
\( f \) is a Maxwellian
iff
\[
\left\{ \frac{\delta f}{\delta t} + \xi \cdot \nabla f \right\} = 0.
\]

Now let us state and prove the celebrated H-Theorem.

**H-**THEOREM: Let \( H = \int_{\mathbb{R}^3} f \log f \ d\xi \).

If \( f \) satisfies the Boltzmann equation, then \( H \) is a monotonically decreasing function of time i.e, 
\[
\frac{dH(t)}{dt} \leq 0.
\]
Proof: The proof follows trivially by the Boltzmann inequality. Consider the Boltzmann equation
\[ \frac{\delta f}{\delta t} + \xi \cdot \nabla_x f = Q(f, f). \]
Multiply both sides of the above equation by \( \log f \) and integrate with respect to \( \xi \). We get
\[ \frac{\delta H}{\delta t} + \nabla_x \cdot \psi = S. \]
where
\[ H = \int_{\mathbb{R}^3} f \log f \, d\xi. \]
\[ \psi = \int_{\mathbb{R}^3} f \xi \log f \, d\xi. \]
\[ S = \int_{\mathbb{R}^3} \log f Q(f, f) \, d\xi. \]
Boltzmann inequality says that \( S \leq 0 \) and \( S = 0 \) iff \( f \) is a Maxwellian. This implies that \( \frac{dH}{dt} \leq 0 \).

\( H \)-functional coincides with the usual entropy of Physicists upto a change of sign. Also it is a dynamical entropy, in the sense that it is defined for non-equilibrium systems. Thus the Botzmann H-Theorem is a manifestation of the Second law of Thermodynamics which states that the physical entropy of an isolated system should not decrease in time. This achievement of producing an analytic proof of the second law for some specific model of statistical mechanics was one of the early goals of Boltzmann and was later considered as one of his most important contributions to Statistical Physics.

But, number of objections were raised against the H-Theorem. The popular paradoxes are Loschmidt’s paradox and Zermelo’s paradox. We will not address these aspects in this essay. A good account about the H-theorem and the irreversibility can be found in [1] and [4].

6 Fluid-Dynamic limit

The concept of Fluid-Dynamic limit is to obtain rigorous derivations of macroscopic models such as fundamental PDEs of fluid mechanics from a microscopic description of matter, say Kinetic theory of gases. The subject of Fluid-Dynamic limit goes back to the work of the founders of the Kinetic theory of gases, J. Clerk Maxwell and L. Boltzmann. D. Hilbert formulated the question of fluid-dynamic limit as a mathematical problem, as an example in his 6th problem on the axiomatization of Physics. Hilbert himself attacked this problem. We shall discuss the method due to Hilbert in this essay.
At first, we shall define the macroscopic quantities like density, bulk velocity and temperature through the velocity distribution function defined on the phase space that solves the Boltzmann Equation. Let us make an other assumption that for a.e $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^3$, $\int_{\mathbb{R}^3} f(t, x, \xi) \, d\xi > 0$.

**Definition:**
Density, $\rho(t, x) = \int_{\mathbb{R}^3} f(t, x, \xi) \, d\xi$.

Bulk velocity, $u(t, x) = \frac{1}{\rho(t, x)} \int_{\mathbb{R}^3} \xi \, f(t, x, \xi) \, d\xi$.

Temperature, $\theta(t, x) = \frac{1}{\rho(t, x)} \int_{\mathbb{R}^3} \frac{1}{3} |\xi - u(t, x)|^2 \, f(t, x, \xi) \, d\xi$.

If $f$ were a Maxwellian, it turns out that the constants $\alpha, \beta$ that appeared in (63) are given by $\beta = \frac{1}{2} \theta$ and $\alpha = \rho (2\pi \theta)^{-\frac{3}{2}}$. The term $u$ in (63) is exactly the bulk velocity. Hence Maxwellian in terms of $\rho, \theta$ and $u$ is given by

$$M = \frac{\rho}{(2\pi \theta)^{\frac{3}{2}}} \exp \left\{ \frac{-|\xi - u|^2}{2\theta} \right\}. \quad (65)$$

Let us now consider the Boltzmann equation

$$\frac{\partial f}{\partial t} + \xi \cdot \nabla_x f = Q(f, f).$$

Multiply the above equation by $\{1, \xi, |\xi|^2\}$ and integrate the resulting equation with respect to $\xi$. This method is called the method of taking moments. Then we will get,

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^3} f \, d\xi + \nabla_x \int_{\mathbb{R}^3} \xi \, f \, d\xi = 0.$$

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^3} \xi \, f \, d\xi + \nabla_x \int_{\mathbb{R}^3} \xi \otimes \xi \, f \, d\xi = 0.$$

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^3} \frac{1}{2} |\xi|^2 \, f \, d\xi + \nabla_x \int_{\mathbb{R}^3} \frac{1}{2} |\xi|^2 \, \xi \, f \, d\xi = 0.$$

The RHS has vanished as $\{1, \xi, |\xi|^2\}$ are collision invariants. If the definitions of $\rho, \theta$ and $u$, the above set of equations become

$$\frac{\partial}{\partial t} \rho + \nabla_x (\rho u) = 0.$$

$$\frac{\partial}{\partial t} (\rho u) + \nabla_x (\rho u \otimes u) + \nabla_x (\rho \theta) = -\nabla_x \int_{\mathbb{R}^3} A(\xi - u) \, f \, d\xi.$$

$$\frac{\partial}{\partial t} \left( \frac{1}{2} \rho |u|^2 + \frac{3}{2} \theta \right) + \nabla_x (\rho u (\frac{1}{2} |u|^2 + \frac{5}{2} \theta)) = -\nabla_x \int_{\mathbb{R}^3} B(\xi - u) \, f \, d\xi - \nabla_x \int_{\mathbb{R}^3} A(\xi - u) \cdot u \, f \, d\xi. \quad (66)$$

where $A$ mentioned here is different form the $A$ appeared in (63). $A$ and $B$ are

$$A(z) = z \otimes z - \frac{1}{3} |z|^2$$

$$B(z) = \frac{1}{2} (|z|^2 - 5) z. \quad (67)$$
The LHS of (66) coincides with that of the compressible euler system with adiabatic exponent $\frac{5}{3}$. The RHS of (66) depends on the solution of the Boltzmann equation $f$ and is in general not determined by $\rho$, $\theta$ and $u$. However, in some limit, it may be possible to approximate the RHS of (66) by appropriate functions of $\rho$, $\theta$ and $u$ there by arriving at a system in closed form with unknowns $(\rho, \theta, u)$. To get euler system in the asymptotic limit of Boltzmann equation, we need to show that the RHS of the 2nd and 3rd equations in (66) vanish in that limit. The problem of finding such closure relations is the key to all the derivations of fluid-dynamic models from the Boltzmann equation.

### 6.1 The scaled Boltzmann Equation

Since we are trying to get the Macroscopic description, we need to scale the Boltzmann equation which is the microscopic description. We shall define the macroscopic space and time variables through the microscopic space and time variables. Before going into the scaling let us define the term Mean free path.

**Definition:** Mean free path is the average distance travelled by a randomly picked gas molecule between two successive collisions.

Let the volume of the container containing gas be $V$. Let the number of gas molecules present in the volume $V$ be $N$. If the $N$ molecules are tightly packed, then the volume occupied by them is called *excluded volume* and is denoted by $V_e$. We expect the Mean free path to be less when there are more number of gas molecules present or when the gas molecules are bigger. So we can use the following approximation for the *mean free path*:

$$
\text{mean free path} \approx \frac{V - V_e}{N A}
$$

where $A$ is the surface area of the section of the particles.

The degree of rarefaction in the gas can be measured by using Knudsen number, $K_n$, defined as the ratio of the mean free path to the size of the container i.e,

$$
K_n = \frac{\text{mean free path}}{\text{macroscopic length scale}}
$$

A rarefied gas is a gas for which $K_n > 1$ while a gas in fluid-dynamic regime satisfies $K_n << 1$.

Let us consider a large container of side $\varepsilon^{-1}$ containing gas where $\varepsilon$ is a parameter to be sent to zero. Let $f^\varepsilon(t, x; \xi)$ be the distribution function of the
particles present in the large container. Let us introduce the new space, time
variables and the density function
\[ r = \varepsilon x, \quad \tau = \varepsilon t, \quad \hat{f}(\tau, r, \xi) = f^\varepsilon(t, x, \xi). \] (70)
The above introduced scaling is called hyperbolic scaling. If the scaling is such
that it is $\varepsilon^{-2}$ in time, then it is called parabolic scaling. The parabolic scaling
is employed in getting the in-compressible Navier-stokes equations. The picture
of the system in terms of the variables $r$ and $\tau$ is called macroscopic whereas
the picture in terms of $x$ and $t$ is called microscopic. It is to be noted that
the mean free path turns out be of the order of $\varepsilon$. If we substitute the new
variables from (70) into the Boltzmann equation, we get the scaled Boltzmann
equation:
\[ \delta \hat{f} + \xi \cdot \nabla_r \hat{f} = \varepsilon^{-1} Q(\hat{f}, \hat{f}). \] (71)
A good presentation of the Scaled Boltzmann Equation can be found in [8]
and [10].

6.2 Linearized collision integral
Let $\rho$ and $\theta > 0$, and pick $u \in \mathbb{R}^3$, the linearization $M_{(\rho, u, \theta)}$ of the Boltzmann
collision integral is defined as follows
\[ L_{M_{(\rho, u, \theta)}} f = -2M_{(\rho, u, \theta)^{-1}} Q(M_{(\rho, u, \theta)}, M_{(\rho, u, \theta)} f) \] (72)
As we are trying to find a solution for the Boltzmann which is close to the
Maxwellian, the above transformation makes sense. We have the general bi-
linear expression associated with $Q(f, f)$ given by
\[ Q(f, h) = \frac{1}{2} \int_{S^2} \int_{R^3} [f' h' + f' h' - fh_\ast - f_\ast h] \ b(g, \omega) \ d\omega \ d\xi. \] (73)
Hence we get
\[ L_{M_{(\rho, u, \theta)}} f = \int_{S^2} \int_{R^3} [f + f_\ast - f' - f'_\ast] M_{(\rho, u, \theta)}(\xi) b(g, \omega) \ d\omega \ d\xi. \] (74)
If we define the scaling transformations of the functions defined on $\mathbb{R}^3$ by
\[ \Lambda_u f(\xi) = f(\xi - u) \quad m_\lambda f(\xi) = \lambda^{-3} f(\frac{\xi}{\lambda}) \]
then we get
\[ M_{(\rho, u, \theta)} = \rho m_{\sqrt{\theta}} \Lambda_u M_{(1, 0, 1)}. \] (75)
We also get that
\[ m_{\sqrt{\theta}} \Lambda_u L_{M_{(1, 0, 1)}}^b f = \rho \theta^2 \Lambda_u L_{M_{(\rho, u, \theta)}}^b (m_{\sqrt{\theta}} \Lambda_u f). \] (76)
Hence it is enough to study the linearization of the collision integral at the Gaussian,

\[ M = M_{(1,0,1)}. \]

Since \[ M_{(1,0,1)} = \frac{1}{(2\pi)^{\frac{3}{2}}} \exp \left\{ \frac{-|\xi|^2}{2} \right\}, \] it greatly simplifies the calculations involved.

We assume that the collision kernel \( b \) satisfies the hard cut-off assumption in the sense of Grad, i.e,

\[ \exists \, \alpha \in [0,1] \text{ and } c_b > 0 \text{ such that for a.e } g \in \mathbb{R}^3 \text{ and } \omega \in S^2, \text{ one has} \]

\[ 0 < b(g,\omega) \leq c_b(1 + |g|^\alpha) \text{ and} \]

\[ \int_{S^2} b(g,\omega) \, d\omega \geq \frac{1}{c_b} (1 + |g|)^\alpha. \]

We shall use \( L \) for \( L^b_M \) from now on. \( L \) can be split as

\[ L(f) = a(|\xi|)f(\xi) - Kf(\xi). \]

where \( a \) is the collision frequency

\[ a(|\xi|) = \int_{S^2} \int_{\mathbb{R}^3} b(\xi - \xi_*,\omega)M(\xi_*)d\xi_*d\omega. \]

\[ K(f) = K_1(f) - K_2(f) \text{ with} \]

\[ K_1(f) = \int_{S^2} \int_{\mathbb{R}^3} f(\xi_*) b(\xi - \xi_*,\omega)M(\xi_*)d\xi_*d\omega. \]

\[ K_2(f) = \int_{S^2} \int_{\mathbb{R}^3} [f(\xi_*) + f(\xi')]b(\xi - \xi_*,\omega)M(\xi_*)d\xi_*d\omega. \]

\( K_2 \) can be further simplified as

\[ K_2(f) = 2 \int_{S^2} \int_{\mathbb{R}^3} f(\xi') b(\xi - \xi_*,\omega)M(\xi_*)d\xi_*d\omega. \]

The above simplification in the hard sphere case was shown by Hilbert. H. Grad showed the above simplification for all cut off potentials that satisfy the Grad cut off condition.

\( K_1 \) is easily seen to be a compact operator on \( L_2(Md\xi) \). A theorem due to Grad says that the operator \( K_2 \) is also compact on \( L_2(Md\xi) \) whenever \( b \) satisfies the Grad cut off condition. The main properties of \( L \) are given in the theorem stated below:

**Theorem:** Assume that the collision kernel \( b \) satisfies the Grad cut off assumption, then \( L \) is an unbounded self-adjoint non-negative Fredholm operator
with domain $L_2(a^2Md\xi)$. Its null space is the space of all collision invariants:

$$\ker L = \text{span}\{1, \xi_1, \xi_2, \xi_3, |\xi|^2\}.$$ 

As a consequence of the above stated theorem, we can say that

$$L(f) = \psi, \quad \psi \in L_2(Md\xi). \quad (77)$$

satisfies the Fredholm Alternative:

(*) either $\psi \in \ker L$, in which case (77) has a unique $\phi_0 \in (L_2(a^2Md\xi) \cap (\ker L)^\perp)$ such that any solution of (77) is of the form $\psi = \psi_0 + \psi_1$ where $\psi_1$ is an arbitrary element of $\ker L$.

(*) or $\psi \in \ker L$, in which case (77) has no solution.

Let us consider the vector field $B$ and the matrix field $A$ that appeared in the RHS of (66). we have,

$$B(\xi) = \frac{1}{2}(|\xi|^2 - 5)\xi, \quad A(\xi) = \xi \otimes \xi - \frac{1}{3}|\xi|^2. \quad (78)$$

It is easy to verify that

$$A_{ij} \in (\ker L)^\perp, \quad B_l \in \ker L, \quad \text{hence} \quad A_{ij} \perp B_l, \quad i, j, l = 1, 2, 3. \quad (79)$$

Attributed to the invariance of $L$ under orthogonal transformation we have

$$\int_{\mathbb{R}^3} A(\xi) f(|\xi|^2) Md\xi = 0. \quad (80)$$

$$\int_{\mathbb{R}^3} B(\xi) f(|\xi|^2) Md\xi = 0. \quad (80)$$

The above results will help us compute the terms in a power series which we are going to consider in the next subsection.

### 6.3 Hilbert Expansion

Let us consider the scaled Boltzmann equation

$$\frac{\delta f^\varepsilon}{\delta t} + \xi \cdot \nabla_x f^\varepsilon = \varepsilon^{-1} Q(f^\varepsilon, f^\varepsilon). \quad (81)$$

We shall assume that the collision kernel $b$ satisfies the Grad cut off assumption. In 1912, Hilbert proposed to find a analytic solution for (81).i.e, he set out to
find a formal power series in $\varepsilon$ as a solution for (81).
Let us try to find a solution of the form

$$f^\varepsilon(t, x, \xi) = \sum_{k=1}^{\infty} \varepsilon^k f^k(t, x, \xi).$$

(82)

By inserting (82) in (81) and matching the various orders in $\varepsilon$, we obtain equations which one can hope to solve recursively:

$$Q(f^0, f^0) = 0.$$  

$$Q(f^1, f^0) = (\delta_t + (\xi \cdot \nabla_x)) f^0.$$  

$$Q(f^2, f^0) = (\delta_t + (\xi \cdot \nabla_x)) f^1 - Q(f^1, f^1).$$

...  

...  

$$Q(f^j, f^1) = (\delta_t + (\xi \cdot \nabla_x)) f^{j-1} - \sum_{k=1}^{j-1} Q(f^i, f^k)$$

with $1 \leq i, k \leq j - 1 : i + k = j$.  

(83)

The first equation of the above set says that $f^0$ is a Maxwellian. Hence there exists $\rho(t, x) > 0, \theta(t, x) > 0$ and $u(t, x) \in \mathbb{R}^3$ such that

$$f^0 = M(\rho, u, \theta).$$

The second equation of (83) can be linearized at the local Maxwellian $f^0$ as

$$L_{f^0} \begin{cases} f^1 \\ f^0 \end{cases} = (\delta_t + (\xi \cdot \nabla_x)) log f^0.$$  

(84)

We now use the tools developed in the previous section. By the Fredholm alternative, (84) has a solution if RHS is orthogonal to $ker L$ in $L_2(Md\xi)$. It turns out that for the solvability of (84) we need to equate some coefficients in the final expression of the RHS of (84) to zero. And those set of equations turn out to be the Euler equations. So, we can solve (84) provided $f^0$ has parameters consistent with the euler equations. We thus have determined a part of $f^1$ that is orthogonal to the nullspace of $L_{f^0}$. The solvability for $f^2$ fully determines $f^1$. More generally, the compatibility condition to guarantee the existence of $f^{n+1}$ provides the system of 5 equations satisfied by $f^n$ which belongs to the nullspace of $L_{f^0}$. The above procedure can obviously be iterated to determine all $f^j$s.
Thus we saw that the leading order term in the expansion is a local Maxwellian state whose parameters are governed by Euler equations of gas dynamics. Obviously a question arises regarding the radius of convergence of the power series considered. This is one of the major drawbacks of this method. As we cannot go on calculating all the \( f^j \)'s, we can truncate the series and add a remainder term to find an approximate solution. A similar idea is used by Caflisch in his theorem where he proved the existence of a solution of an initial value problem for Boltzmann Equation that is close to the local Maxwellian.

### 6.4 The compressible Euler limit

Let us consider the scaled Boltzmann Equation:

\[
\frac{\delta f^\varepsilon}{\delta t} + \xi \cdot \nabla_x f^\varepsilon = \varepsilon^{-1} Q(f^\varepsilon, f^\varepsilon).
\]

Consider the compressible Euler system for a perfect monatomic gas:

\[
\begin{align*}
\delta_t \rho + \text{div}_x (\rho u) &= 0, \\
\delta_t (\rho u) + \text{div}_x (\rho u \otimes u) + \nabla_x (\rho \theta) &= 0, \\
\delta_t (\rho (\frac{1}{2} |u|^2 + \frac{5}{2} \theta)) &= 0, \\
(\rho, u, \theta)|_{t=0} &= (\rho^{in}, u^{in}, \theta^{in}).
\end{align*}
\]

Let \((\rho, u, \theta)\) be the solution of (85) with a lifespan \(T > 0\). Define the local Maxwellian from the above mentioned \(\rho, u\) and \(\theta\) as \(E(t, x, \xi) = M(\rho, u, \theta)\). A theorem due to Caflisch succeeded in getting a solution for the Boltzmann equation which tends to the local Maxwellian in the limit. Here we just state the theorem without proof.

**Caflisch’s theorem**: There exists \(\varepsilon_0\) such that for each \(\varepsilon \in (0, \varepsilon_0)\), there is a unique solution \(f^\varepsilon\) of the Boltzmann equation on \([0, T) \times \mathbb{R}^6\) satisfying the estimate

\[
\sup_{0 \leq t \leq T} ||f(t, ., .) - E(t, ., .)||_{L^2_\mathbb{R}^6} = O(\varepsilon) \quad \text{as} \quad \varepsilon \rightarrow 0^+.
\]

The above theorem says that as the mean free path tends to zero, the solutions of the Boltzmann equation approach the local Maxwellian. We already know that the moments of the Maxwellians yield the compressible Euler equations. Hence the Euler equations are got from the Boltzmann equation in the above mentioned limit.
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