6. TWO-SCALE CONVERGENCE METHOD

In this chapter, we revert back to the periodic setting, i.e., we will be treating elliptic boundary value problems with periodically oscillating coefficients. In Lemma A.17 and Lemma A.18 (see Appendix on Basic functional analysis), we remark that the classical weak $L^p$ convergences fail to capture any oscillations in the dilated sequences. To illustrate, let us consider the periodic function $\sin(2n\pi x)$ for the values $n = 1$ and $n = 10$.

![Plot of $\sin(2n\pi x)$ on $[-3, 3]$ for $n = 1$ (left); $n = 10$ (right)](image)

Figure 5. Plot of $\sin(2n\pi x)$ on $[-3, 3]$ for $n = 1$ (left); $n = 10$ (right)

As is illustrated in Figure 5, larger the value of the parameter $n$ is, greater the oscillations are. This is typical of dilated sequences of periodic functions. In Lemma A.17, we stated that for any $Y$-periodic function $f(x)$, the associated sequence of dilated functions $f^\varepsilon(x) = f\left(\frac{x}{\varepsilon}\right)$ has the following weak convergence property:

$$f^\varepsilon\left(\frac{x}{\varepsilon}\right) \rightarrow (f) := \int_Y f(y) \, dy \quad \text{weakly in } L^p(\Omega).$$

Remark that the weak limit fails to capture the high frequency oscillations in the dilated sequence.

In the late 80’s, a new notion of weak convergence was developed which overcame this difficulty. This is the theory of \textit{two-scale convergence}. In fact, this theory develops some analytical tools with the help of some functional spaces. The function spaces which play a crucial role are

$$L^p(\Omega; C_{\text{per}}(Y)) := \left\{ f(x, y) \text{ measurable s.t. } \int_{Y} \sup_{y \in Y} |f(x, y)|^p \, dy < \infty \right\}.$$ 

\begin{definition}[Two-scale convergence] A family $\{u^\varepsilon(x)\} \subset L^p(\Omega)$ with $p \in (1, \infty)$ is said to two-scale converge to $u_0(x,y) \in L^p(\Omega \times Y)$ if for every test function $\psi(x,y) \in L^p'(\Omega; C_{\text{per}}(Y))$ the following limit holds:

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} u^\varepsilon(x) \psi\left(x, \frac{y}{\varepsilon}\right) \, dx = \int_{\Omega} \int_{Y} u_0(x,y)\psi(x,y) \, dy \, dx.$$ 

The above convergence will be denoted hereafter as $u^\varepsilon(x) \overset{\text{2-scale}}{\rightharpoonup} u_0(x,y).$
\end{definition}
Let us first remark that, if there is a two-scale limit to a given sequence, then that two-scale limit is unique. On the contrary, let us suppose that for a sequence \( \{u^{\varepsilon}\} \), there are two two-scale limits \( u_0(x, y) \) and \( \tilde{u}_0(x, y) \), i.e.,

\[
\lim_{\varepsilon \to 0} \int_{\Omega} u^{\varepsilon}(x) \psi \left( x, \frac{x}{\varepsilon} \right) \, dx = \int_{\Omega} \int_{Y} u_0(x, y) \psi(x, y) \, dy \, dx
\]

\[
\lim_{\varepsilon \to 0} \int_{\Omega} u^{\varepsilon}(x) \psi \left( x, \frac{x}{\varepsilon} \right) \, dx = \int_{\Omega} \int_{Y} \tilde{u}_0(x, y) \psi(x, y) \, dy \, dx
\]

Subtracting one for the other, we get

\[
\int_{\Omega} \int_{Y} \left( u_0(x, y) - \tilde{u}_0(x, y) \right) \psi(x, y) \, dy \, dx = 0
\]

for any arbitrary \( \psi(x, y) \in L^{\infty}(\Omega; \mathbb{C}_{\text{per}}(Y)) \). Hence, we have

\[
u_0(x, y) = \tilde{u}_0(x, y) \quad \text{for a.e.} \quad (x, y) \in \Omega \times Y.
\]

Hence the uniqueness.

Next, we state and prove an important result on the test functions.

**Lemma 6.2 (Test functions lemma).** Let \( f(x, y) \in L^1(\Omega; \mathbb{C}_{\text{per}}(Y)) \). Then

- For each \( \varepsilon > 0 \), the dilated function \( f \left( x, \frac{x}{\varepsilon} \right) \) is measurable.
- We have the inequality

\[
\left\| f \left( \cdot, \frac{\cdot}{\varepsilon} \right) \right\|_{L^1(\Omega)} \leq \left\| f \right\|_{L^1(\Omega; \mathbb{C}_{\text{per}}(Y))}
\]

- We further have the limit behaviour

\[
\lim_{\varepsilon \to 0} \int_{\Omega} f \left( \cdot, \frac{\cdot}{\varepsilon} \right) \, dx = \int_{\Omega} \int_{Y} f(x, y) \, dy \, dx.
\]

**Proof.** We will not be proving the assertion about measurability of the dilated sequence of the locally periodic test function.

Taking \( n \in \mathbb{N} \), let us partition the unit cube \( Y \) into little cubes \( \{Y_i\}_{i=1}^{n^d} \) such that

\[
|Y_i| = \frac{1}{n^d} \quad \text{for each} \quad i \in \{1, \ldots, n^d\}
\]

\[
|Y_i \cap Y_j| = 0 \quad \text{if} \quad i \neq j
\]

\[
Y = \bigcup_{i=1}^{n^d} Y_i
\]

Let us denote by \( \chi_i(y) : Y \mapsto \{0, 1\} \), the characteristic function of the set \( Y_i \), i.e.,

\[
\chi_i(y) = \begin{cases} 
1 & \text{if} \quad y \in Y_i \\
0 & \text{otherwise}
\end{cases}
\]

Let us extend the above characteristic function to the whole of \( \mathbb{R}^d \) by \( Y \)-periodicity. We abuse the notation by denoting the extension as \( \chi_i \). Note that \( \chi_i \in L^\infty(\mathbb{R}^d) \).

Let us consider the dilatations of the characteristic function as

\[
\chi_i^{\varepsilon}(x) := \chi_i \left( \frac{x}{\varepsilon} \right).
\]
From Lemma A.17, it follows that
\[ \chi_i \rightharpoonup \chi \] weakly * in \( L^\infty(\mathbb{R}^d) \).

Our strategy to prove (62) for each \( f \in L^1(\Omega; C_{per}(Y)) \) is to first prove (62) for a particular class of step functions. Define a step function
\[
f_n(x, y) := n \sum_{i=1}^d f(x, y_i)\chi_i(y)
\]
with \( y_i \in Y_i \) being some arbitrary point. Observe that
\[
\lim_{\varepsilon \to 0} \int_{\Omega} f_n \left( x, \frac{x}{\varepsilon} \right) \, dx = \sum_{i=1}^d \lim_{\varepsilon \to 0} \int_{\Omega} f(x, y_i)\chi_i \left( \frac{x}{\varepsilon} \right) \, dx
\]
where we used the fact that \( f \) is an \( L^1 \) function in the \( x \) variable and that \( \chi_i \left( \frac{x}{\varepsilon} \right) \) weak * converges to its average over the unit cube. Hence have proved that
\[
\lim_{\varepsilon \to 0} \int_{\Omega} f_n \left( x, \frac{x}{\varepsilon} \right) \, dx = \int_{\Omega} \int_{Y} f_n(x, y) \, dy \, dx.
\]

Let us now consider the difference
\[
\left| \int_{\Omega} f \left( x, \frac{x}{\varepsilon} \right) \, dx - \int_{\Omega} \int_{Y} f(x, y) \, dy \, dx \right| \leq \left| \int_{\Omega} f \left( x, \frac{x}{\varepsilon} \right) \, dx - \int_{\Omega} f_n \left( x, \frac{x}{\varepsilon} \right) \, dx \right|
\]
\[
+ \left| \int_{\Omega} f_n \left( x, \frac{x}{\varepsilon} \right) \, dx - \int_{\Omega} \int_{Y} f_n(x, y) \, dy \, dx \right|
\]
\[
+ \left| \int_{\Omega} \int_{Y} f_n(x, y) \, dy \, dx - \int_{\Omega} \int_{Y} f(x, y) \, dy \, dx \right|
\]
Passing to the limit as \( \varepsilon \to 0 \), we observe that the second term on the right hand side of the above inequality vanishes as we have already proved that the result holds for step functions. Taking supremum in the \( y \) variable in the remaining two terms on the right hand side of the above inequality results in
\[
\lim_{\varepsilon \to 0} \left| \int_{\Omega} f \left( x, \frac{x}{\varepsilon} \right) \, dx - \int_{\Omega} \int_{Y} f(x, y) \, dy \, dx \right| \leq 2 \int_{\Omega} \sup_{y \in Y} \left| f(x, y) - f_n(x, y) \right| \, dx.
\]

Goal is to show that
\[
\| f_n - f \|_{L^1(\Omega; C_{per}(Y))} \to 0 \quad \text{as} \quad n \to \infty.
\]

Define
\[
g_n(x) := \sup_{y \in Y} \left| f_n(x, y) - f(x, y) \right|.
\]

As \( f(x, y) \) is continuous in the \( y \) variable, we have:
\[
g_n(x) \to 0 \quad \text{for a.e.} \quad x \in \Omega \quad \text{as} \quad n \to \infty.
\]
Moreover
\[ g_n(x) \leq 2 \sup_{y \in Y} |f(x, y)| \in L^1(\Omega). \]
Lebesgue dominated convergence theorem implies
\[ \|g_n\|_{L^1(\Omega; C_{\text{per}}(Y))} = \|f_n - f\|_{L^1(\Omega; C_{\text{per}}(Y))} \to 0 \quad \text{as } n \to \infty. \]
Hence the result. \(\square\)

**Remark 6.3.** Take \( f \in L^p(\Omega; C_{\text{per}}(Y)) \). Then, apply the result of Lemma 6.2 to the function \( |f(x, y)|^p \) yields
\[ \lim_{\varepsilon \to 0} \int_{\Omega} |f(x, \frac{x}{\varepsilon})|^p \, dx = \int_{\Omega} \int_Y |f(x, y)|^p \, dy \, dx. \]

**Remark 6.4.** Take a smooth function \( a(x, y) \) which is \( Y \)-periodic in the \( y \) variable. Consider the following dilated sequence
\[ a^\varepsilon(x) := a \left( x, \frac{x}{\varepsilon} \right). \]
Then, for a test function \( \psi(x, y) \), we can treat the product \( a(x, y) \psi(x, y) \) as a function in \( L^1(\Omega; C_{\text{per}}(Y)) \). Hence, an application of Lemma 6.2 implies that
\[ \lim_{\varepsilon \to 0} \int_{\Omega} a \left( x, \frac{x}{\varepsilon} \right) \psi \left( x, \frac{x}{\varepsilon} \right) \, dx = \int_{\Omega} \int_Y a(x, y) \psi(x, y) \, dy \, dx, \]
i.e.,
\[ a^\varepsilon \xrightarrow{2\text{-scale}} a(x, y). \]

**Remark 6.5.** Consider a family of functions defined as a power series
\[ v^\varepsilon(x) = v_0 \left( x, \frac{x}{\varepsilon} \right) + \varepsilon v_1 \left( x, \frac{x}{\varepsilon} \right) + \varepsilon^2 v_2 \left( x, \frac{x}{\varepsilon} \right) + \ldots \]
with smooth coefficients functions \( v_i(x, y) \) which are \( Y \)-periodic in the \( y \) variable. It follows from Remark 6.4 that
\[ v^\varepsilon \xrightarrow{2\text{-scale}} v_0(x, y). \]
This observation has an important implication in connection to the two-scale asymptotic expansions method that we employed in Chapter 2 as a homogenization tool. Loosely speaking, the two-scale theory is inspired by the two-scale expansions method and it "justifies" the formal expansion made in Chapter 2.

**Remark 6.6.** Consider a function \( a(x, y) \) which is \( Y \)-periodic in the \( y \) variable. Consider the dilated sequence defined as
\[ a^\varepsilon(x) := a \left( x, \frac{x}{\varepsilon^2} \right). \]
Note that the dilatations are at a different scale (of \( O(\varepsilon^{-2}) \)) compared to the dilatations used in Remark 6.4. To compute the two-scale limit associated with the above defined dilated sequence \( a^\varepsilon(x) \), we need to study
\[ \lim_{\varepsilon \to 0} \int_{\Omega} a \left( x, \frac{x}{\varepsilon^2} \right) \psi \left( x, \frac{x}{\varepsilon} \right) \, dx. \]
Before we proceed to determine the above limit, let us take $Y, Z$ to denote two copies of the unit cube $[0,1]^d$. Take a function $f(x,y,z)$ which is $Y$-periodic in the $y$ variable and is $Z$-periodic in the $z$ variable. Consider the dilated sequence defined as

$$f^\varepsilon(x) := f \left( x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2} \right).$$

A result similar in spirit to Lemma A.18 yields the following weak limit:

$$f^\varepsilon \rightharpoonup \int_Y \int_Z f(x,y,z) \, dz \, dy$$

weakly in $L^p(\Omega)$ with $p \in [1, \infty)$ and weakly * in $L^\infty(\Omega)$.

Let us now get back to the limit (63). Let us treat the product $a(x,z)\psi(x,y)$ as a function of $x, y$ and $z$ variables which is $Z$-periodic in the $z$ variable and $Y$-periodic in the $y$ variable. Taking the characteristic function $\chi(x)$ as an $L^1$-test function in the above mentioned $L^\infty$ weak * convergence, we have

$$\lim_{\varepsilon \to 0} \int_{\Omega} a \left( x, \frac{x}{\varepsilon^2} \right) \psi \left( x, \frac{x}{\varepsilon} \right) \chi(x) \, dx = \int_{\Omega} \int_{Y} \int_{Z} a(x,z)\psi(x,y) \, dz \, dy \, dx,$$

which essentially says that

$$a \left( x, \frac{x}{\varepsilon^2} \right) \xrightarrow{2\text{-scale}} \int_{Z} a(x,z) \, dz.$$

This observation alludes to the fact that two-scale convergence can only pick up those oscillations which are in resonance with the oscillations in the test functions, i.e., if the oscillations in the sequence are not in-sync with those in the test functions, then the notion of two-scale convergence is as good as the classical weak $L^p$ convergence.

**Proposition 6.7.** Suppose the family $\{u^\varepsilon(x)\} \subset L^p(\Omega)$ is such that

$$u^\varepsilon \rightharpoonup u_0$$

strongly in $L^p(\Omega)$.

Then

$$u^\varepsilon \xrightarrow{2\text{-scale}} u_0.$$

**Proof.** Take $\psi \in L^\infty(\Omega; C_\text{per}(Y))$ and consider the difference

$$\left| \int_{\Omega} u^\varepsilon(x)\psi \left( x, \frac{x}{\varepsilon} \right) \, dx - \int_{\Omega} \int_{Y} u_0(x)\psi(x,y) \, dy \, dx \right|$$

$$\leq \left| \int_{\Omega} \left( u^\varepsilon(x) - u_0(x) \right)\psi \left( x, \frac{x}{\varepsilon} \right) \, dx \right|$$

$$+ \left| \int_{\Omega} u_0(x)\psi \left( x, \frac{x}{\varepsilon} \right) \, dx - \int_{\Omega} \int_{Y} u_0(x)\psi(x,y) \, dy \, dx \right|$$

$$=: I_1 + I_2.$$
By Hölder inequality, we have
\[
\left| \int_\Omega \left( u^\varepsilon(x) - u_0(x) \right) \psi \left( x, \frac{x}{\varepsilon} \right) \, dx \right| \leq \left( \int_\Omega |u^\varepsilon(x) - u_0(x)|^p \, dx \right)^{\frac{1}{p}} \left( \int_\Omega \left| \psi \left( x, \frac{x}{\varepsilon} \right) \right|^q \, dx \right)^{\frac{1}{q}}^{\frac{1}{p}}
\]
\[
\leq \| u^\varepsilon - u_0 \|_{L^p(\Omega)} \| \psi \|_{L^q(\Omega; C_{\text{per}}(Y))}.
\]
Because of the strong convergence in $L^p(\Omega)$ of $\{u^\varepsilon\}$, it follows that
\[
\lim_{\varepsilon \to 0} I^\varepsilon_1 = 0.
\]
Furthermore, we have that the test function $\psi \in L^q(\Omega; C_{\text{per}}(Y))$ and that the limit point $u_0 \in L^p(\Omega)$. Hence the product $u_0(x)\psi(x, y) \in L^1(\Omega; C_{\text{per}}(Y))$. So, by Lemma 6.2, it follows that
\[
\lim_{\varepsilon \to 0} I^\varepsilon_2 = 0.
\]
Hence the following difference vanishes in the $\varepsilon \to 0$ limit
\[
\left| \int_\Omega u^\varepsilon(x) \psi \left( x, \frac{x}{\varepsilon} \right) \, dx - \int_\Omega \int_Y u_0(x) \psi(x, y) \, dy \, dx \right|.
\]
Hence the result. $\Box$

In the above proposition, we proved that a strongly converging sequence two-scale converges to the same limit. More importantly, the above result says that there are is no second scale to pick-up in a strongly converging sequence. Next, we explore the link between two-scale convergence and the classical weak $L^p$ convergence.

**Proposition 6.8.** Suppose the family $\{u^\varepsilon(x)\} \subset L^p(\Omega)$ is such that
\[ u^\varepsilon \overset{2\text{-scale}}{\rightarrow} u_0(x, y). \]
Then
\[ u^\varepsilon \rightarrow \int_Y u_0(x, y) \, dy \text{ weakly in } L^p(\Omega). \]

**Proof.** By Definition 6.1, we have that
\[
\lim_{\varepsilon \to 0} \int_\Omega u^\varepsilon(x) \psi \left( x, \frac{x}{\varepsilon} \right) \, dx = \int_\Omega \int_Y u_0(x, y) \psi(x, y) \, dy \, dx
\]
for all $\psi(x, y) \in L^q(\Omega; C_{\text{per}}(Y))$. In particular, the above limit holds for $\psi(x, y) \equiv \phi(x)$ with $\phi(x) \in L^p(\Omega)$, i.e., for those test functions that are independent of the $y$ variable. Making this choice in the earlier expression yields
\[
\lim_{\varepsilon \to 0} \int_\Omega u^\varepsilon(x) \phi(x) \, dx = \int_\Omega \left( \int_Y u_0(x, y) \, dy \right) \phi(x) \, dx.
\]
Hence the result. $\Box$

The following result demonstrates further links between weak $L^p$ convergence and the two-scale convergence of Definition 6.1.
Proposition 6.9. Suppose \( \{u^\varepsilon\} \subset L^p(\Omega) \) be such that
\[
u^\varepsilon \overset{\text{2-scale}}{\rightharpoonup} u_0(x, y).
\]
Then
\[
\lim_{\varepsilon \to 0} \|u^\varepsilon\|_{L^p(\Omega)} \geq \|u_0\|_{L^p(\Omega \times Y)} \geq \|\pi\|_{L^p(\Omega)}
\]
where \( \pi(x) \) is the weak \( L^p \)-limit:
\[
\pi(x) := \int_Y u_0(x, y) \, dy.
\]
Proof. Let us first prove the far right end of the inequality (64). By Hölder inequality, we have
\[
|\pi(x)| = \left| \int_Y u_0(x, y) \, dy \right| \leq \left( \int_Y |u_0(x, y)|^p \, dy \right)^{\frac{1}{p}} \left( \int_Y 1^q \, dy \right)^{\frac{1}{q}}
\]
Taking the \( p \)-th power of the above inequality and integrating over \( \Omega \) yields
\[
\int_\Omega |\pi(x)|^p \, dx \leq \int_\Omega \int_Y |u_0(x, y)|^p \, dy \, dx.
\]
Now, let us get to proving the far left end of the inequality (64). To begin with, let us remark that \( u_0 \in L^p(\Omega \times Y) \) implies that \( |u_0|^{p-2} u_0 \in L^{p'}(\Omega \times Y) \). The proof goes via the use of smooth approximating functions. More precisely, pick a family \( \{\varepsilon_n(x, y)\} \subset L^{p'}(\Omega; C_{\text{per}}(Y)) \) such that
\[
\lim_{n \to \infty} \|\varepsilon_n - |u_0|^{p-2} u_0\|_{L^{p'}(\Omega \times Y)} = 0
\]
which in turn implies that
\[
\lim_{n \to \infty} \int_\Omega \int_Y |\varepsilon_n(x, y)|^{p'} \, dy \, dx = \int_\Omega |u_0(x, y)|^{p'} \, dy \, dx.
\]
Hölder inequality says that for \( f \in L^p(\Omega) \) and \( g \in L^{p'}(\Omega) \), we have
\[
\int f(x) g(x) \, dx \leq \left( \int |f(x)|^p \, dx \right)^{\frac{1}{p}} \left( \int |g(x)|^{p'} \, dx \right)^{\frac{1}{p'}}
\]
Young’s inequality says that for \( a, b \geq 0 \), we have
\[
ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'} = \frac{a^p}{p} + \frac{b^{p'}}{p'}
\]
Using the Young’s inequality in the above Hölder inequality results in
\[
\int_\Omega f(x) g(x) \, dx \leq \frac{1}{p} \int_\Omega |f(x)|^p \, dx + \frac{1}{p'} \int_\Omega |g(x)|^{p'} \, dx.
\]
Note that we have
\[
\frac{p}{p'} = p - 1.
\]
Hence we rewrite (65) as
\[ \int_\Omega |f(x)|^p \, dx \geq p \int_\Omega f(x)g(x) \, dx - (p-1) \int_\Omega |g(x)|^p \, dx. \]
In (66), we make the following choices for $f$ and $g$:
\[ f(x) = u^\varepsilon(x); \quad g(x) = \psi_n \left( x, \frac{x}{\varepsilon} \right). \]
The inequality of interest becomes
\[ \int_\Omega |u^\varepsilon(x)|^p \, dx \geq p \int_\Omega u^\varepsilon(x) \psi_n \left( x, \frac{x}{\varepsilon} \right) \, dx - (p-1) \int_\Omega |\psi_n \left( x, \frac{x}{\varepsilon} \right)|^p \, dx. \]
Passing to the limit as $\varepsilon \to 0$ in the above inequality yields
\[ \lim_{\varepsilon \to 0} \int_\Omega |u^\varepsilon(x)|^p \, dx \geq p \int_\Omega \int_Y |u_0(x,y)\psi_n(x,y)| \, dy \, dx - (p-1) \int_\Omega \int_Y |u_0(x,y)|^p \, dy \, dx. \]
where we used the fact that $u^\varepsilon$ $2\text{-scale} \to u_0(x,y)$ and the result of Lemma 6.2 for the function $\psi_n \in L^{p'}(\Omega, C_0(Y))$. Passing to the limit as $n \to \infty$ in the inequality (67) yields
\[ \lim_{\varepsilon \to 0} \int_\Omega |u^\varepsilon(x)|^p \, dx \geq p \int_\Omega \int_Y |u_0(x,y)|^p \, dy \, dx - (p-1) \int_\Omega \int_Y |u_0(x,y)|^p \, dy \, dx. \]
This proves the far left end of the inequality (64). \(\square\)

The next result (given without proof) concerns passing to the limit in a product of two sequences which converge in the sense of two-scale.

**Proposition 6.10.** Let $\{u^\varepsilon\} \subset L^p(\Omega)$ be a family such that
\[ u^\varepsilon \overset{2\text{-scale}}{\rightharpoonup} u_0(x,y). \]
Further assume that
\[ \lim_{\varepsilon \to 0} \|u^\varepsilon\|_{L^p(\Omega \times Y)} = \|u_0\|_{L^p(\Omega \times Y)}. \]
Let $\{v^\varepsilon\} \subset L^{p'}(\Omega)$ be another family such that
\[ v^\varepsilon \overset{2\text{-scale}}{\rightharpoonup} v_0(x,y). \]
Then we have the following convergence property in the $\varepsilon \to 0$ limit:
\[ u^\varepsilon v^\varepsilon \overset{\text{in the sense of distributions}}{\longrightarrow} \int_Y u_0(x,y)v_0(x,y) \, dy \]
i.e.,
\[ \lim_{\varepsilon \to 0} \int_\Omega u^\varepsilon(x)v^\varepsilon(x)\varphi(x) \, dx = \int_\Omega \left( \int_Y u_0(x,y)v_0(x,y) \, dy \right) \varphi(x) \, dx \]
for all $\varphi \in C_0^{\infty}(\Omega)$.

So far, we have explored some properties of the notion of two-scale convergence. We have not demonstrated any compactness result for the two-scale convergence defined in Definition 6.1.
Theorem 6.11. Suppose \( \{u^\varepsilon\} \subset L^p(\Omega) \) be a family with \( p \in (1, \infty) \) be such that
\[
\|u^\varepsilon\|_{L^p(\Omega)} \leq C
\]
for some constant \( C \) independent of \( \varepsilon \). Then, there exists a subsequence (still indexed \( u^\varepsilon \)) and there exists a limit \( u_0(x, y) \in L^p(\Omega \times Y) \) such that
\[
\lim_{n \to \infty} \|u^\varepsilon - u_n\|_{L^p(\Omega \times Y)} = 0,
\]
where we have used the uniform boundedness of the family \( \{u^\varepsilon\} \) and there exists a limit \( u_0(x, y) \in L^p(\Omega \times Y) \) such that
\[
\lim_{n \to \infty} \|u^\varepsilon - u_n\|_{L^p(\Omega \times Y)} = 0.
\]
Before we get to the proof of the above theorem, let us recall some facts from functional analysis which will play a crucial role in the proof of Theorem 6.11.

- Consider the Banach space \( L^p(\Omega; B) \) with \( p \in [1, \infty) \) where \( B \) is a Banach space. If \( B \) is separable (i.e., it has a countable dense subset), then so is the function space \( L^p(\Omega; B) \).
- Banach-Alaoglu theorem: Let \( B \) be a separable Banach space. Let \( B' \) denote the associated dual space. Let \( \{f_n\}_{n=1}^\infty \subset B' \) be a uniformly bounded sequence, i.e.,
\[
\|f_n\|_{B'} \leq C
\]
with the constant \( C \) being independent of \( n \). Then, there exists a subsequence (still indexed \( \{f_n\} \)) such that the extract subsequence converges in the corresponding weak * topology, i.e., there exists a \( f_\infty \in B' \) such that
\[
\lim_{n \to \infty} \langle f_n, \varphi \rangle_{B', B} \to \langle f_\infty, \varphi \rangle_{B', B}
\]
for all \( \varphi \in B \), where \( \langle \cdot, \cdot \rangle_{B', B} \) denotes duality product between \( B' \) and \( B \).
- Riesz representation theorem: Let \( \mathcal{H} \) be a Hilbert space. Let \( \mathcal{H}' \) denote the associated dual space. Let \( F \in \mathcal{H}' \). Then, there exists a unique \( f \in \mathcal{H} \) such that
\[
F(v) = \langle f, v \rangle_{\mathcal{H}}
\]
where \( \langle \cdot, \cdot \rangle_{\mathcal{H}} \) denotes the inner product in the Hilbert space \( \mathcal{H} \).

Proof of Theorem 6.11. We give the proof in the case \( p = 2 \). Following arguments can be adapted to the case \( p \neq 2 \). Taking \( \psi(x, y) \in L^2(\Omega; C_{per}(Y)) \), let us define
\[
\mathcal{F}^\varepsilon(\psi) := \int_{\Omega} u^\varepsilon(x) \psi \left( x, \frac{x}{\varepsilon} \right) \, dx.
\]
Cauchy-Schwarz inequality yields
\[
|\mathcal{F}^\varepsilon(\psi)| \leq \left( \int_{\Omega} |u^\varepsilon(x)|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} \left| \psi \left( x, \frac{x}{\varepsilon} \right) \right|^2 \, dx \right)^{\frac{1}{2}}
\leq C \|\psi\|_{L^2(\Omega; C_{per}(Y))}
\]
where we have used the uniform boundedness of the family \( \{u^\varepsilon\} \) and the assertion of Lemma 6.2. Hence the family \( \{\mathcal{F}^\varepsilon\} \) forms a bounded linear form on \( L^2(\Omega; C_{per}(Y)) \). Furthermore, the family \( \{\mathcal{F}^\varepsilon\} \subset (L^2(\Omega; C_{per}(Y)))' \) forms a uniformly bounded family. Hence, invoking Banach-Alaoglu theorem, there exists a subsequence (still indexed \( \{\mathcal{F}^\varepsilon\} \)) and there exists a \( \mathcal{F}^0 \in (L^2(\Omega; C_{per}(Y)))' \) such that as \( \varepsilon \to 0 \),
\[
\mathcal{F}^\varepsilon(\psi) \longrightarrow \mathcal{F}^0(\psi) \quad \text{for all } \psi \in L^2(\Omega; C_{per}(Y)).
\]
Let us recall that $L^2(\Omega; C\text{per}(Y))$ is dense in $L^2(\Omega \times Y)$, i.e., for each $\psi(x,y) \in L^2(\Omega \times Y)$, there exists a sequence $\{\psi_n\}_{n=1}^{\infty} \subset L^2(\Omega; C\text{per}(Y))$ such that
\begin{equation}
\lim_{n \to \infty} \|\psi_n - \psi\|_{L^2(\Omega \times Y)} = 0.
\end{equation}

For any given $\psi \in L^2(\Omega \times Y)$, let us define the extension of $\mathcal{F}^0$ as
\[ \mathcal{F}^0(\psi) = \lim_{n \to \infty} \mathcal{F}^0(\psi_n) \]
where $\{\psi_n\}$ is as in (68). Furthermore, $\mathcal{F}^0$ is in the dual of the Hilbert space $L^2(\Omega \times Y)$. Hence, by the Riesz representation theorem, there exists a unique $u_0 \in L^2(\Omega \times Y)$ such that
\[ \mathcal{F}^0(\psi) = \mathcal{F}^0(\psi) = \int_{\Omega} \int_{Y} u_0(x,y) \psi(x,y) \, dy \, dx \quad \forall \psi \in L^2(\Omega \times Y). \]

In particular, for $\psi \in L^2(\Omega; C\text{per}(Y))$, we have
\[ \mathcal{F}^0(\psi) = \int_{\Omega} \int_{Y} u_0(x,y) \psi(x,y) \, dy \, dx, \]
because $\mathcal{F}^0$ is the extension of $\mathcal{F}^0$. Thus, we have proved the existence of a $u_0(x,y) \in L^2(\Omega \times Y)$ such that
\[ \lim_{\varepsilon \to 0} \mathcal{F}^\varepsilon(\psi) = \lim_{\varepsilon \to 0} \int_{\Omega} u_\varepsilon^\varepsilon(x,y) \psi \left( x, \frac{x}{\varepsilon} \right) \, dx = \int_{\Omega} \int_{Y} u_0(x,y) \psi(x,y) \, dy \, dx \]
for all $\psi \in L^2(\Omega; C\text{per}(Y))$. Hence the result. \hfill \Box

Having proved a compactness result for the two-scale convergence, we try to understand the consequences of having an additional regularity on the function sequence. This is the object of the next theorem.

**Theorem 6.12.** Suppose $\{u^\varepsilon\} \subset H^1(\Omega)$ be a family such that
\[ u^\varepsilon \rightharpoonup \overline{u}(x) \quad \text{weakly in } H^1(\Omega). \]

Then, there exists a subsequence (still indexed $u^\varepsilon$) and there exists a function $u_1(x,y) \in L^2(\Omega; H^1_{\text{per}}(Y)/R)$ such that
\[ u^\varepsilon \rightharpoonup_{2\text{-scale}} \overline{u}(x) \]
\[ \nabla u^\varepsilon \rightharpoonup_{2\text{-scale}} \nabla_y \overline{u}(x) + \nabla_x u_1(x,y). \]

**Proof.** We have that the family $\{u^\varepsilon\}$ satisfies
\[ \|u^\varepsilon\|_{H^1(\Omega)} \leq C \]
which in particular says that
\[ \|u^\varepsilon\|_{L^2(\Omega)} \leq C \quad \text{and} \quad \|\nabla u^\varepsilon\|_{L^2(\Omega)} \leq C. \]
Hence, invoking Theorem 6.11 for the sequences \( \{u^\varepsilon\} \) and \( \{\nabla u^\varepsilon\} \), we get the existence of two-scale limits \( u_0(x,y) \in L^2(\Omega \times Y) \) and \( \zeta_0(x,y) \in (L^2(\Omega \times Y))^d \) such that

\[
\begin{align*}
u^\varepsilon \xrightarrow{\text{2-scale}} u_0(x,y) \\
\nabla u^\varepsilon \xrightarrow{\text{2-scale}} \zeta_0(x,y).
\end{align*}
\]

Taking a smooth compactly supported \( Y \)-periodic (in the \( y \) variable) vector-valued function \( \Psi(x,y) \), consider

\[
\int_\Omega \nabla u^\varepsilon(x) \cdot \Psi(x,\frac{x}{\varepsilon}) \, dx = -\int_\Omega u^\varepsilon(x) \text{div}_x \Psi(x,\frac{x}{\varepsilon}) \, dx - \frac{1}{\varepsilon} \int_\Omega u^\varepsilon(x) \text{div}_y \Psi(x,\frac{x}{\varepsilon}) \, dx
\]

where we have performed an integration by parts. Multiplying the previous expression by \( \varepsilon \) and passing to the limit as \( \varepsilon \to 0 \) (where we use the information that \( u_0(x,y) \) is the two-scale limit of the family \( \{u^\varepsilon\} \)) yields

\[
\int_\Omega \int_Y u_0(x,y) \text{div}_y \psi(x,y) \, dy \, dx = 0
\]

which implies that \( u_0 \) is a function independent of the \( y \) variable. From Proposition 6.8, it follows that

\[
u_0(x,y) \equiv \pi(x).
\]

Next we wish to characterise the two-scale limit \( \zeta_0(x,y) \). To that end, let us consider the test function \( \Psi(x,y) \) as above. We further assume that \( \text{div}_y \Psi(x,y) = 0 \). Thus, we have

\[
\int_\Omega \nabla u^\varepsilon(x) \cdot \Psi(x,\frac{x}{\varepsilon}) \, dx = -\int_\Omega u^\varepsilon(x) \text{div}_x \Psi(x,\frac{x}{\varepsilon}) \, dx.
\]

Passing to the limit in the previous expression yields

\[
\int_\Omega \int_Y \zeta_0(x,y) \cdot \Psi(x,y) \, dy \, dx = -\int_\Omega \int_Y \pi(x) \text{div}_x \Psi(x,y) \, dy \, dx.
\]

Performing an integration by parts in the previous expression yields

\[
\int_\Omega \int_Y \left( \zeta_0(x,y) - \nabla_y \pi(x) \right) \cdot \Psi(x,y) \, dy \, dx = 0.
\]

Thus, we have shown that the function \( \zeta_0(x,y) - \nabla_y \pi(x) \) is orthogonal (in \( L^2(\Omega \times Y) \)) to divergence free functions (in the \( y \) variable). Hence this function should be a gradient in the \( y \) variable. So, there exists a function \( u_1(x,y) \in L^2(\Omega; H^1_{\text{per}}(Y)) \) such that

\[
\zeta_0(x,y) - \nabla_y \pi(x) = \nabla_y u_1(x,y).
\]

This proves the result. \( \square \)

Next, we state some results which can be proved via density arguments (proofs not given here).

**Lemma 6.13.** Suppose \( \{u^\varepsilon\} \subset L^p(\Omega) \) is a family such that

\[
u^\varepsilon \xrightarrow{\text{2-scale}} u_0(x,y).
\]
Then, for any \( \psi(x, y) \in L^p_{\text{per}}(Y; \mathbb{C}(\overline{Y})) \), we have

\[
\lim_{\varepsilon \to 0} \int_{\Omega} u^\varepsilon(x) \psi \left( x, \frac{x}{\varepsilon} \right) \, dx = \int_{\Omega} \int_{Y} u_0(x, y) \psi(x, y) \, dy \, dx.
\]

Note that, unlike the choice of test functions in Definition 6.1, the choice of test function in the above lemma is such that the test functions are continuous in the \( x \) variable. The above lemma helps us handle \( L^\infty \) coefficients in the homogenization context which is made precise in the following remark.

**Remark 6.14.** Take \( a(y) \in L^\infty_{\text{per}}(Y) \). Then, notice that

\[ a(y) \psi(x, y) \in L^p_{\text{per}}(Y; \mathbb{C}(\overline{Y})) \quad \forall \psi(x, y) \in L^p_{\text{per}}(Y; \mathbb{C}(\overline{Y})). \]

Hence if

\[ u^\varepsilon \to \text{2-scale} u_0(x, y), \]

then

\[
\lim_{\varepsilon \to 0} \int_{\Omega} u^\varepsilon(x) a \left( \frac{x}{\varepsilon} \right) \psi \left( x, \frac{x}{\varepsilon} \right) \, dx = \int_{\Omega} \int_{Y} u_0(x, y) a(y) \psi(x, y) \, dy \, dx.
\]

**Remark 6.15.** Suppose \( \{u^\varepsilon\} \subset L^2(\Omega) \) be such that there exists \( u_0(x, y) \in L^2(\Omega \times Y) \) and

\[ u^\varepsilon \to \text{2-scale} u_0(x, y). \]

Then, for all \( \psi(x, y) \) of the form

\[ \psi(x, y) = \psi_1(x) \psi_2(y) \]

with \( \psi_1 \in L^{2s}(\Omega), \psi_2 \in L^{2t}_{\text{per}}(Y) \) with \( 1 \leq s, t \leq \infty \) and \( \frac{1}{s} + \frac{1}{t} = 1 \), we have

\[
\lim_{\varepsilon \to 0} \int_{\Omega} u^\varepsilon(x) \psi_1(x) \psi_2 \left( \frac{x}{\varepsilon} \right) \, dx = \int_{\Omega} \int_{Y} u_0(x, y) \psi_1(x) \psi_2(y) \, dy \, dx.
\]